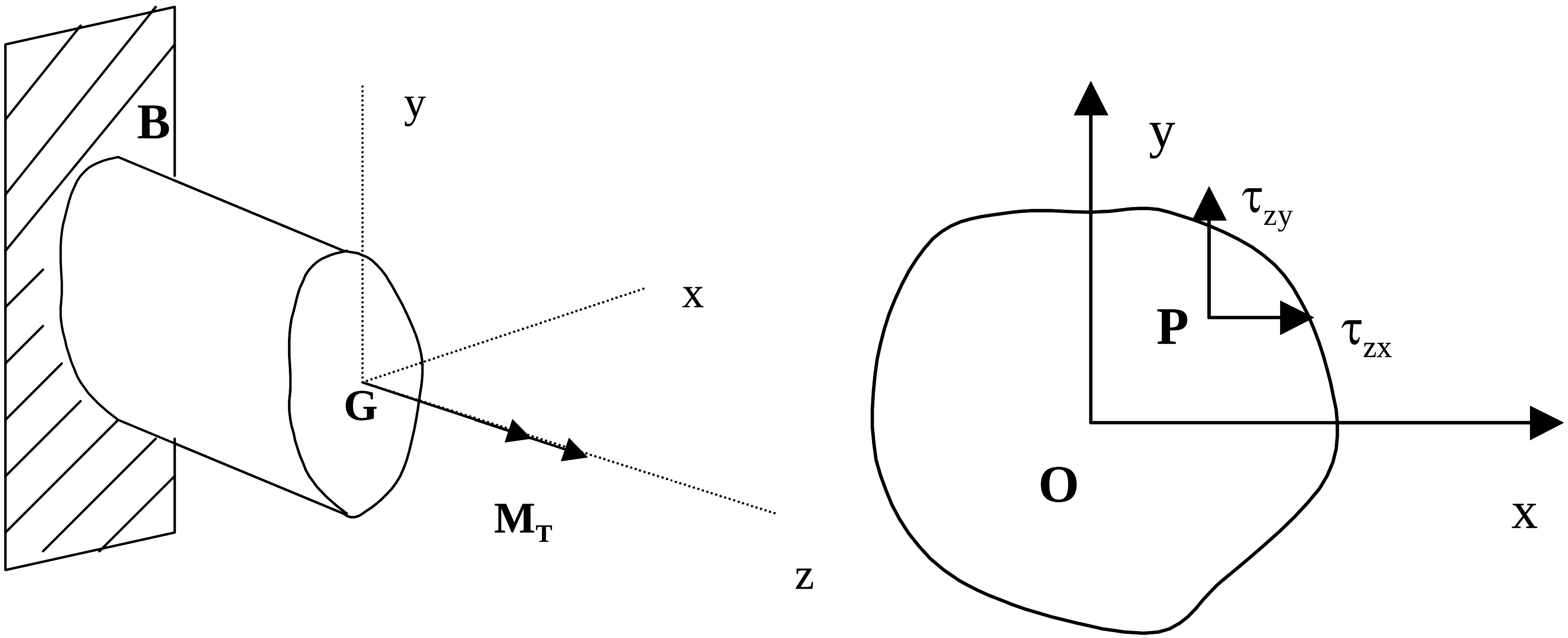
CAPÍTULO 12

TORSIÓN



Solución tensional:

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \qquad \tau_{zx} \neq 0 \quad \tau_{zy} \neq 0$$

¿Verifica esa solución tensional las ecuaciones de equilibrio interno?

1) $$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \tau_{zx}}{\partial z} = 0 \qquad \tau_{zx} = \tau_{zx}(x, y)$$

2) $$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \tau_{zy}}{\partial z} = 0 \qquad \tau_{zy} = \tau_{zy}(x, y)$$

FUNCION DE TORSION

$$\boxed{\Phi = \Phi(x, y)}$$

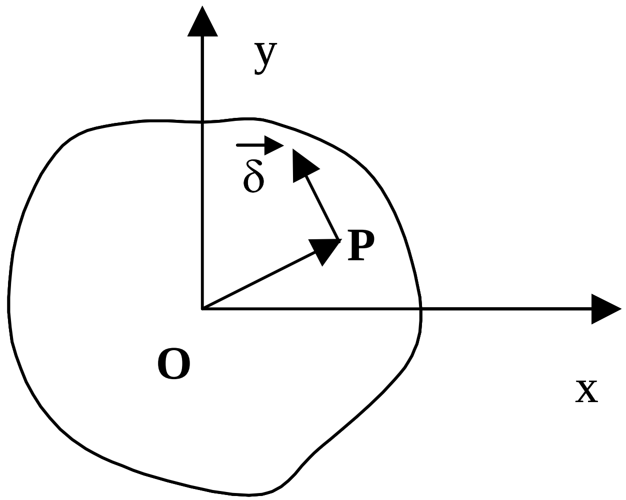
$$\tau_{zx} = \frac{\partial \Phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial \Phi}{\partial x}$$

3) $$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial x \partial y} = 0$$

Ecuaciones de Beltrami:

I_1^{σ} NULO

$$\left.\begin{aligned} (1+\nu)\Delta\tau_{zx} + \frac{\partial I_1^{\sigma}}{\partial z \partial x} = 0 \quad &\Rightarrow \quad \Delta\tau_{zx} = 0 \\ (1+\nu)\Delta\tau_{zy} + \frac{\partial I_1^{\sigma}}{\partial z \partial y} = 0 \quad &\Rightarrow \quad \Delta\tau_{zy} = 0 \end{aligned}\right\} \Delta\Phi = constante$$

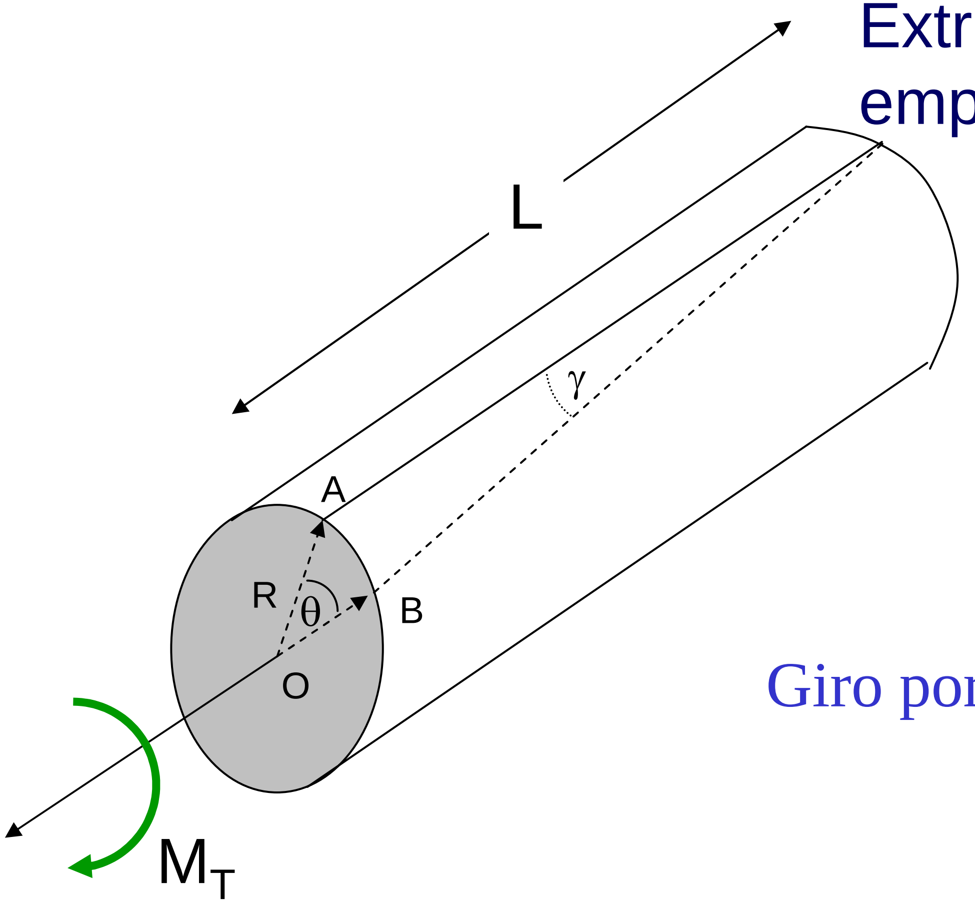


z = distancia de la sección considerada a la que tomamos de referencia y que supondremos que no gira en su plano

Supongamos que el giro unitario (ω en rad/m) permanece constante a lo largo de la pieza (lo cual es cierto si la pieza es recta, de sección constante, y se aplican momentos torsores iguales y opuestos en sus dos caras extremas)

El giro absoluto experimentado por la sección genérica (a distancia *z* de la de referencia) alrededor del punto O (que no gira) sería igual al producto $\omega . z$

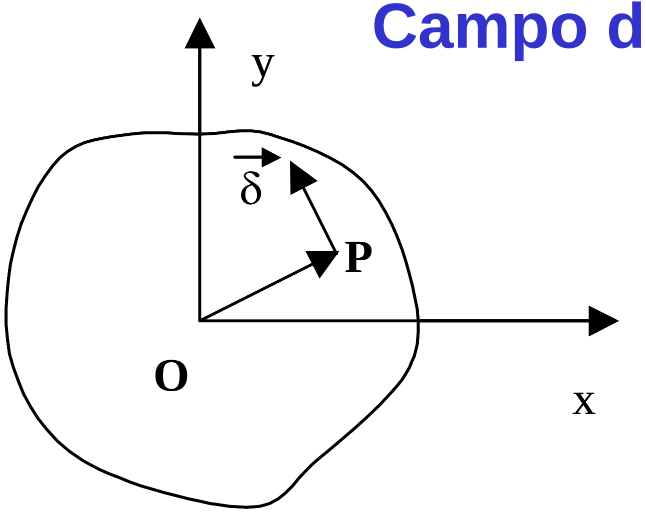
$$\theta = \omega . z$$



Giro por unidad de longitud:

$$\omega=\theta/L$$

Campo de desplazamientos:



$$u = -(\omega.z)y$$
$$v = (\omega.z)x$$
$$w = \omega W_1(x, y)$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] = \left[\omega \frac{\partial W_1}{\partial x} - \omega\, y\right] \qquad \gamma_{zy} = \frac{\tau_{zy}}{G} = \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] = \left[\omega \frac{\partial W_1}{\partial y} + \omega\, x\right]$$

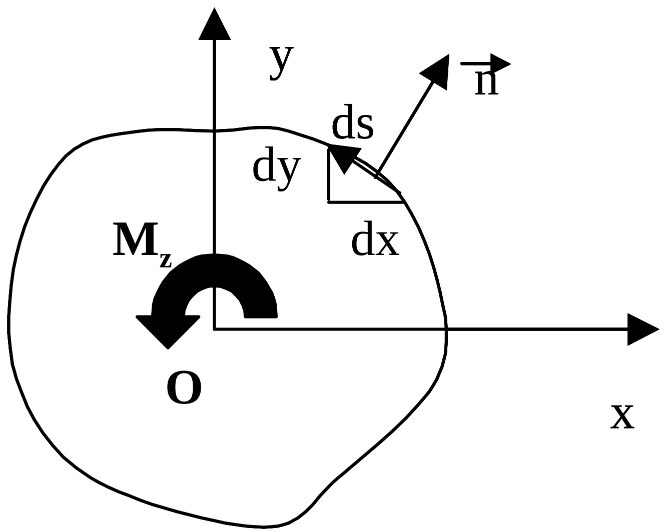
$$\Delta\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right)$$

$$\Delta\Phi = \frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x}$$

$$\boxed{\Delta\Phi} = G\left[\omega \frac{\partial^2 W_1}{\partial x \partial y} - \omega - \omega \frac{\partial^2 W_1}{\partial x \partial y} - \omega\right] = \boxed{-2G\omega}$$

$$\Delta\Phi = -2G\omega$$

Equilibrio en el contorno:



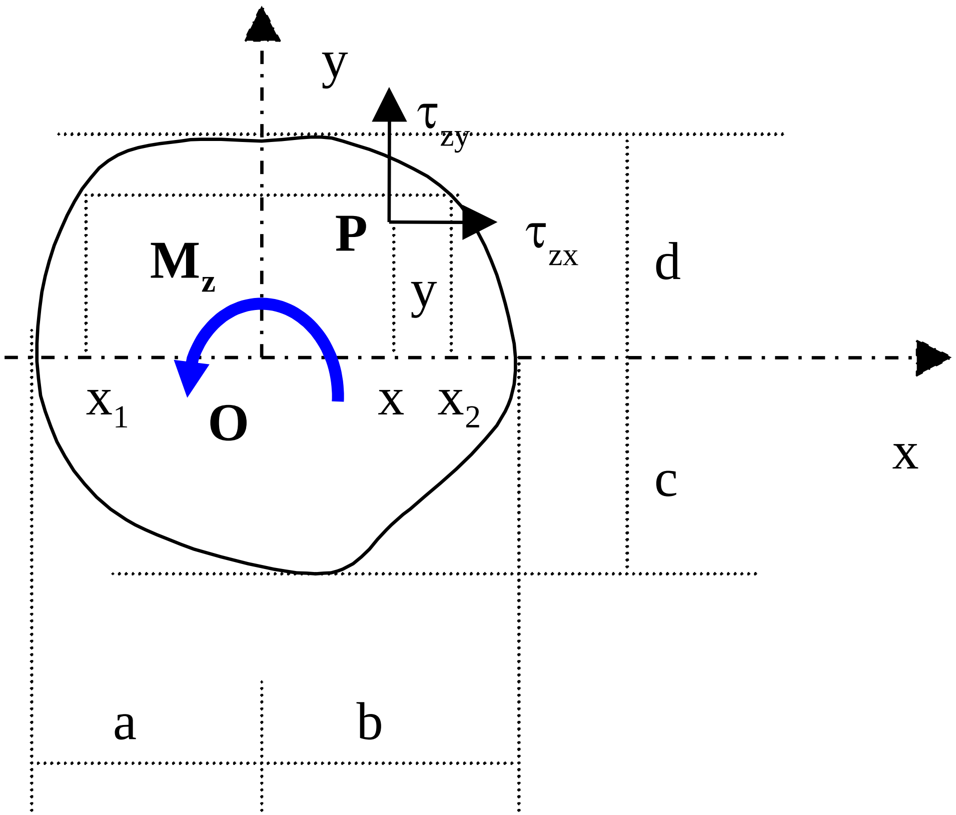
$$l\tau_{zx} + m\tau_{zy} = 0$$

$$l = \frac{dy}{ds} \qquad m = -\frac{dx}{ds}$$

$$\frac{dy}{ds}\frac{\partial \Phi}{\partial y} + \frac{dx}{ds}\frac{\partial \Phi}{\partial x} = 0$$

$$\boxed{\frac{\partial \Phi}{\partial s} = 0}$$

Relación entre el momento torsor y la función de torsión:



$$M_z = \iint_{\Omega} \left(\tau_{zy} x - \tau_{zx} y \right) d\Omega$$

$$\tau_{zx} = \frac{\partial \Phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial \Phi}{\partial x}$$

$$M_z = 2 \iint_{\Omega} \Phi(x, y) dx.dy$$

En resumen:

Un problema de torsión se resuelve obteniendo la función de torsión ***Φ(x,y)*** del problema, que debe poseer las siguientes propiedades:

$$\Delta\Phi = -2G\omega$$

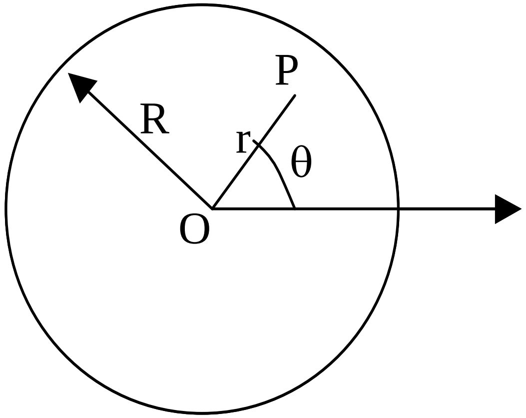
$$\frac{\partial\Phi}{\partial s} = 0$$

$$M_z = 2\iint_{\Omega} \Phi(x, y)dx.dy$$

Conocida la función de torsión ***Φ(x,y)***, las componentes no nulas de la tensión tangencial son:

$$\tau_{zx} = \frac{\partial\Phi}{\partial y} \qquad \tau_{zy} = -\frac{\partial\Phi}{\partial x}$$

TORSIÓN EN PIEZAS DE SECCIÓN CIRCULAR



Ecuación de la circunferencia:

$$r^2 - R^2 = 0$$

$$f(r) = r^2 - R^2$$

$$\Delta f = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} = 2 + 2 = 4 = constante$$

Función de torsión:

$$\Phi = C\left(r^2 - R^2\right)$$

$$\Phi = C\left(r^2 - R^2\right)$$

$$\Delta\phi = 4C = -2G\omega \Rightarrow C = -\frac{G\omega}{2}$$

$$\Phi = -\frac{G\omega}{2}\left(r^2 - R^2\right)$$

$$M_z = 2\iint_{\Omega} \Phi \, d\Omega = -G\omega\int_0^{2\pi} d\theta\int_0^{R}\left(r^2 - R^2\right) r \, dr = \frac{\pi R^4}{2} G\omega = I_0 G\omega$$

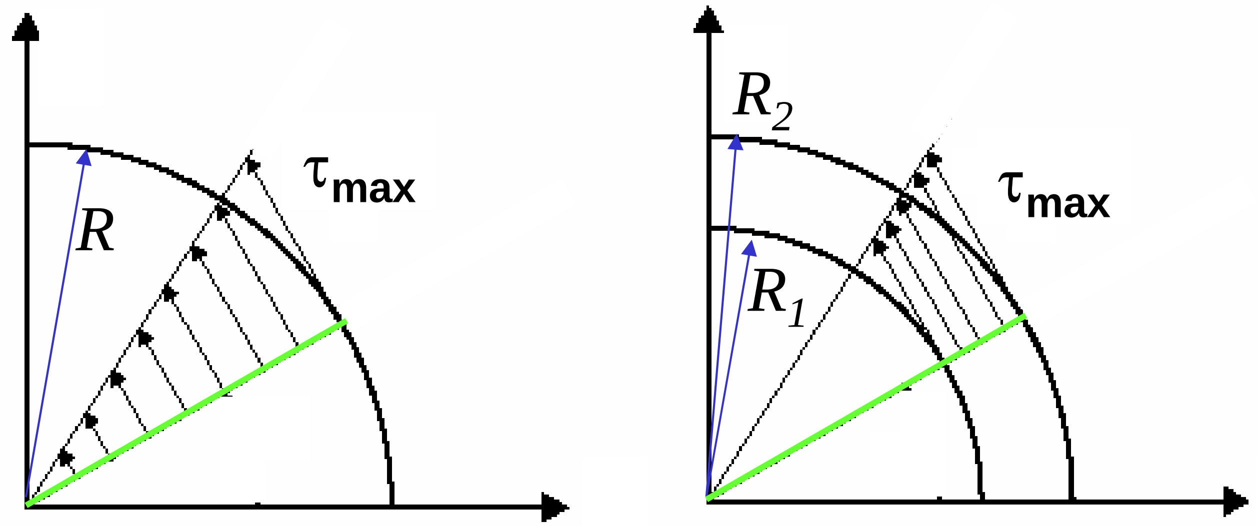
$$\Phi = -\frac{M_z}{2I_0}\left(r^2 - R^2\right)$$

Como: $r^2 = x^2 + y^2$

$$\left.\begin{aligned} \tau_{zx} &= \frac{\partial \Phi}{\partial y} = -\frac{M_z}{I_0} y \\ \tau_{zy} &= -\frac{\partial \Phi}{\partial x} = \frac{M_z}{I_0} x \end{aligned}\right\} \quad \tau = \frac{M_z}{I_0} r$$

$$\boxed{\tau_{max} = \frac{M_z}{I_0} R}$$

$$I_O = \frac{\pi R^4}{2}$$



Sección circular maciza

Sección tubular gruesa

$$I_O = \frac{\pi}{2} R^4$$

$$I_O = \frac{\pi}{2}\left(R_2^4 - R_1^4\right)$$

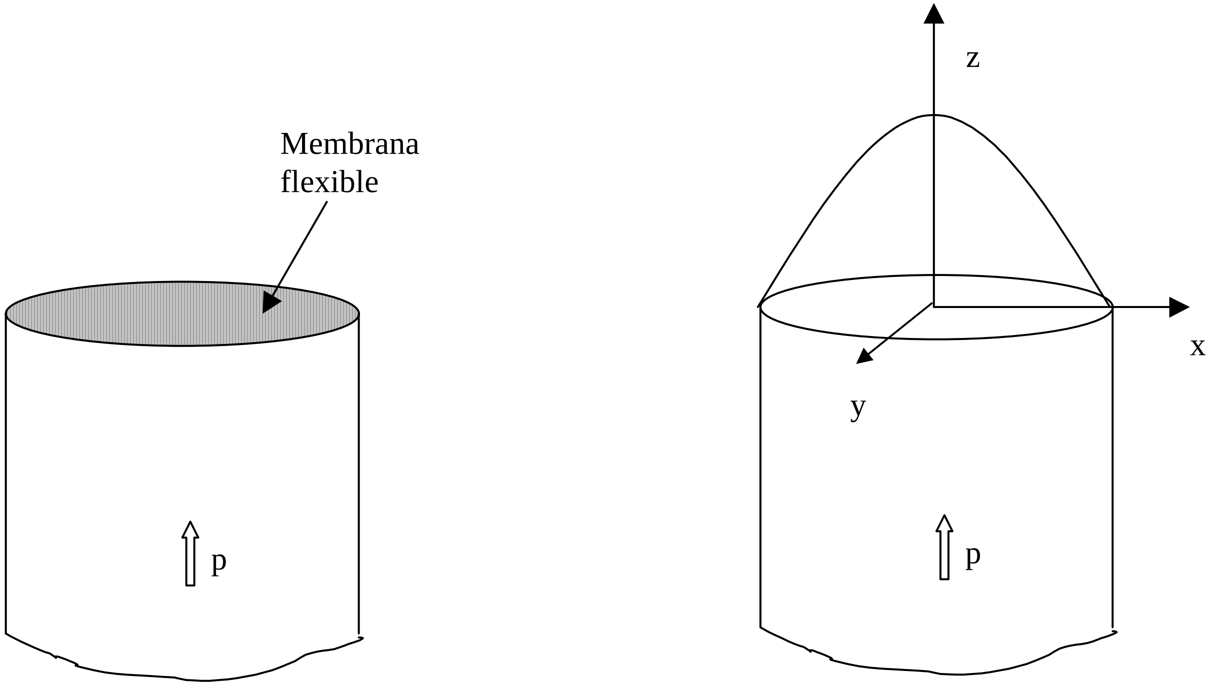
$$\tau = \frac{M_z}{I_0} r$$

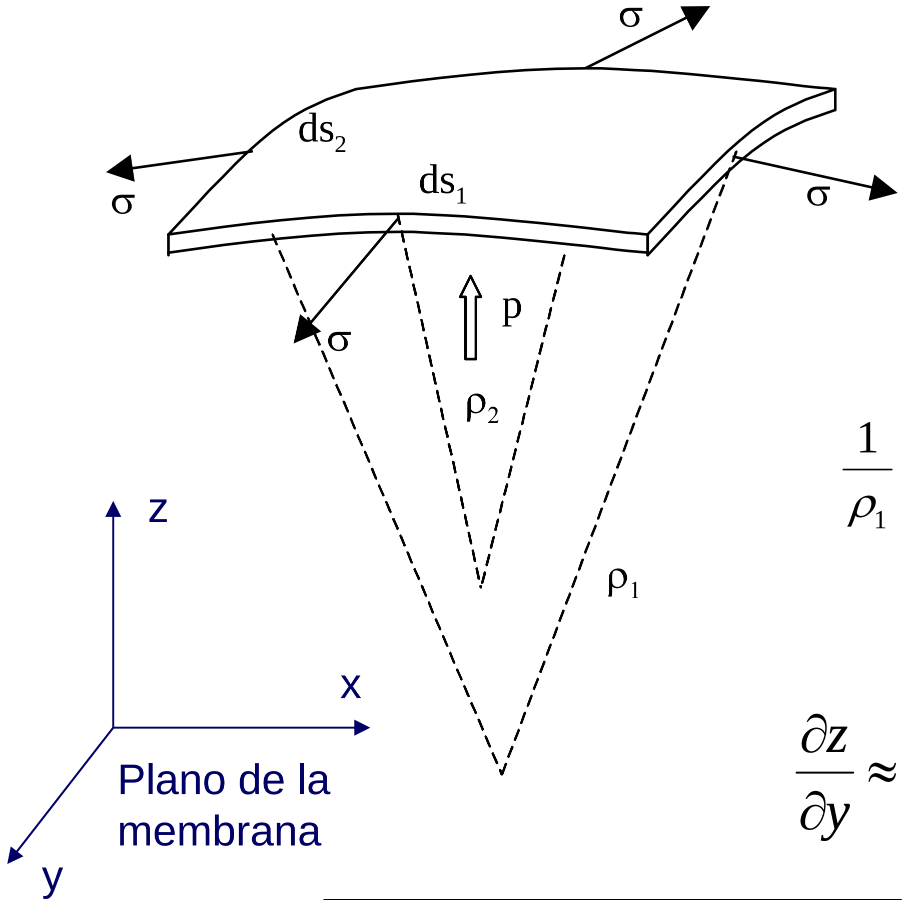
TORSIÓN EN PIEZAS DE SECCIÓN RECTANGULAR

$$\tau_{max} = \frac{M_z}{c_1 a b^2} \qquad \phi = \frac{M_z L}{c_2 a b^3 G}$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

ANALOGIA DE LA MEMBRANA





$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=-\frac{p}{\sigma}$$

$$\frac{1}{\rho_1}=\frac{\dfrac{\partial^2 z}{\partial y^2}}{\left[\sqrt{1+\left(\dfrac{\partial z}{\partial y}\right)^2}\,\right]^3}\qquad \frac{1}{\rho_2}=\frac{\dfrac{\partial^2 z}{\partial x^2}}{\left[\sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2}\,\right]^3}$$

$$\frac{\partial z}{\partial y}\approx\frac{\partial z}{\partial x}\approx 0 \quad\Rightarrow\quad \frac{1}{\rho_1}=\frac{\partial^2 z}{\partial y^2} \quad ,, \quad \frac{1}{\rho_2}=\frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\Delta z=-\frac{p}{\sigma}$$

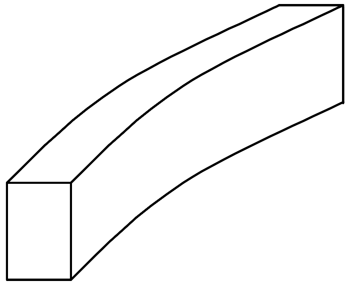
$${\partial z}/{\partial s}=0$$ en el contorno

PROBLEMA DE TORSIÓN	PROBLEMA DE LA MEMBRANA
$\Delta\Phi = -2G\omega$	$\Delta z = -\dfrac{p}{\sigma}$
a lo largo del contorno $\partial\Phi / \partial s = 0$	a lo largo del contorno $\partial z / \partial s = 0$

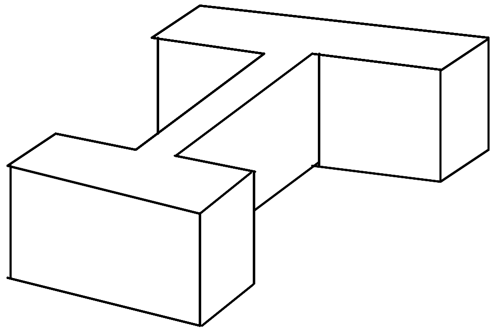
$$\frac{\Delta\Phi}{-2G\omega} = 1 = \frac{\Delta z}{-\frac{p}{\sigma}} \quad \Rightarrow \quad \Delta\Phi = \frac{2G\omega}{p/\sigma}\Delta z \quad \Rightarrow \quad \boxed{\Phi = \frac{2G\omega}{p/\sigma}\, z}$$

TORSION EN PIEZAS DE SECCION DE PARED DELGADA

Perfiles abiertos

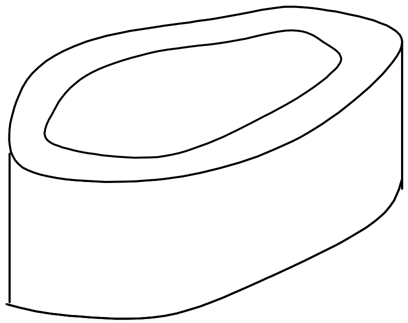


Sin ramificar

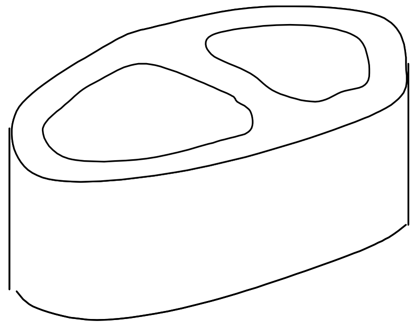


Ramificado

Perfiles cerrados

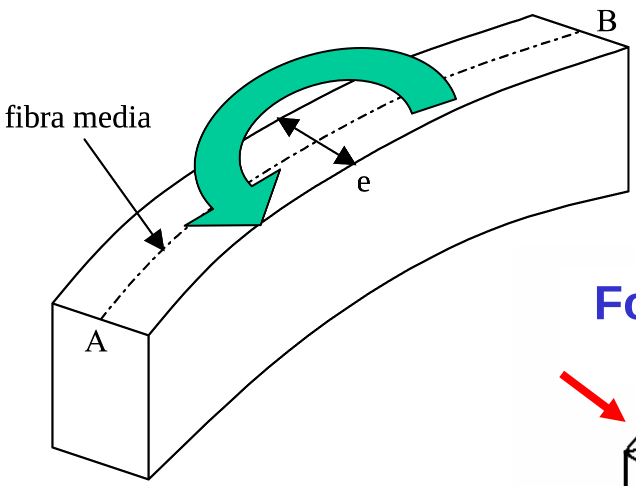


De una célula

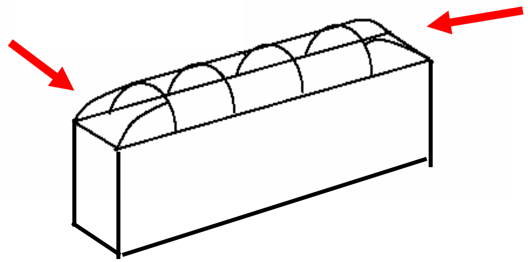


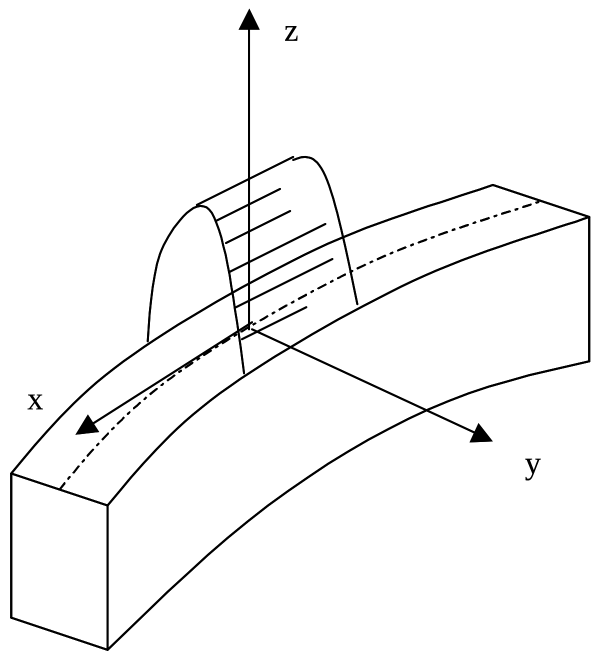
De dos células

PERFILES ABIERTOS SIN RAMIFICAR



Forma de la membrana:





$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{\sigma} \qquad \frac{\partial^2 z}{\partial x^2} = 0 \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{p}{\sigma} \qquad z = -\frac{p}{\sigma}\frac{y^2}{2} + C_1$$

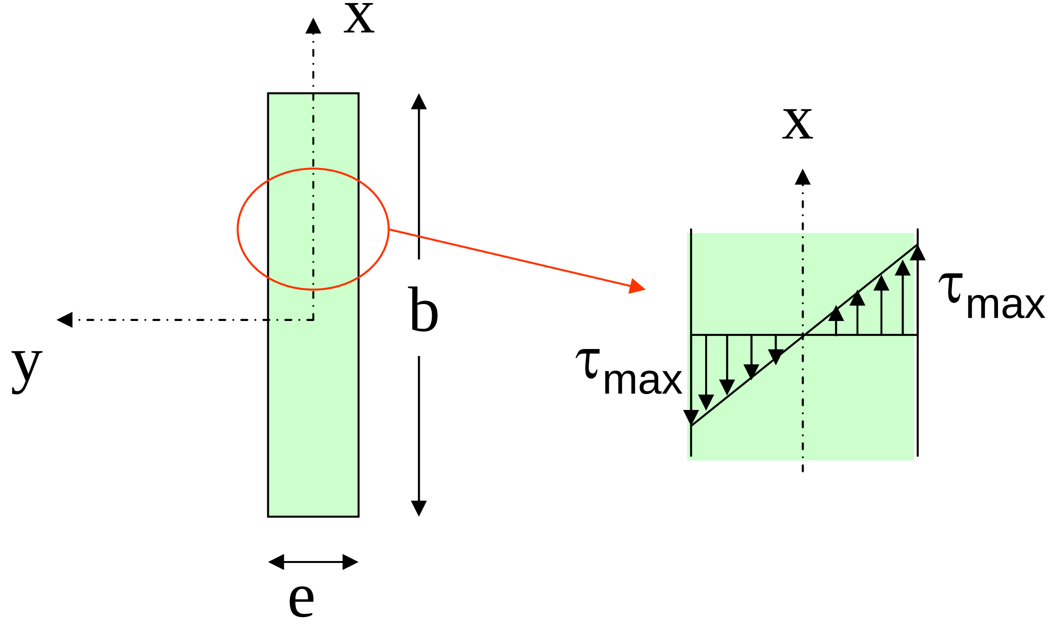
$z=0$ cuando $y=e/2$ ⇨

$$z = \frac{p}{2\sigma}\left(\frac{e^2}{4} - y^2\right)$$

$$\Phi = \frac{2G\omega}{p/\sigma} z = G\omega\left(\frac{e^2}{4} - y^2\right)$$

$$\tau_{zx} = \frac{\partial \Phi}{\partial y} = -2G\omega\, y$$

$$\tau_{zy} = 0$$



$$\tau_{\max} = G\omega e$$

$$M_z = 2\iint_{\Omega} \Phi(x,y)\,d\Omega = \frac{4G\omega}{p/\sigma}\iint_{\Omega} z\,d\Omega = \frac{4G\omega}{p/\sigma}\int_{-\frac{e}{2}}^{\frac{e}{2}} dy\left(\frac{e^2}{4} - y^2\right)\cdot\frac{p}{2\sigma}\int_0^l dx$$

Si el espesor de la sección es constante y de valor e:

$$M_z = G\omega l\frac{e^3}{3} = \tau_{max}\frac{e^2 l}{3} \Rightarrow \tau_{max} = \frac{3M_z}{e^2 l}$$

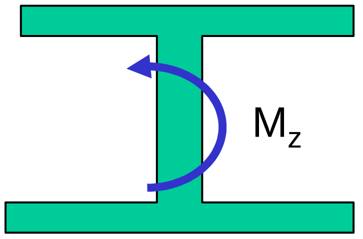
Módulo de torsión:

$$K = \frac{M_z}{G\omega}$$

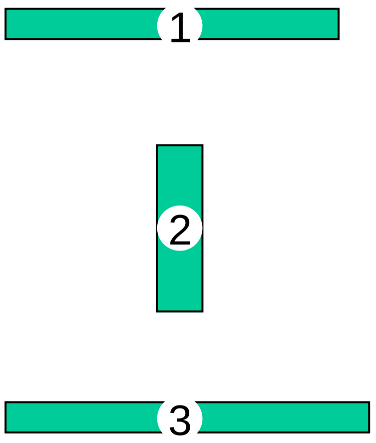
Para el caso de una sección abierta de espesor constante:

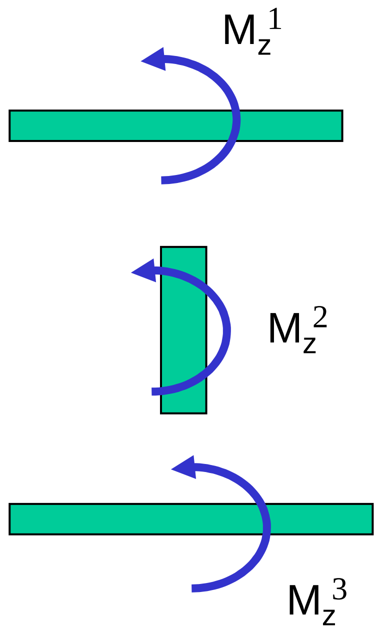
$$K = \frac{1}{3}e^3 l$$

PERFILES ABIERTOS RAMIFICADOS



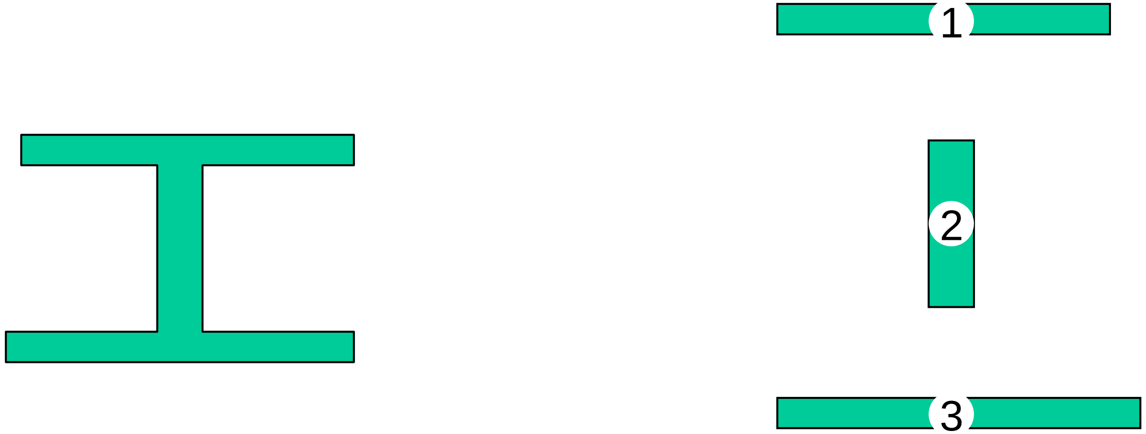
***M_z* = momento torsor aplicado a una sección de perfil abierto ramificado**





M_z^i = momento torsor absorbido por el perfil abierto *i*

$$M_z = \sum_{i=1}^{n} M_z^i$$



$$\omega_1 = \omega_2 = \ldots\ldots\ldots\ldots = \omega_n = \omega$$

$$G\omega_1 = G\omega_2 = \ldots\ldots\ldots\ldots = G\omega_n = G\omega$$

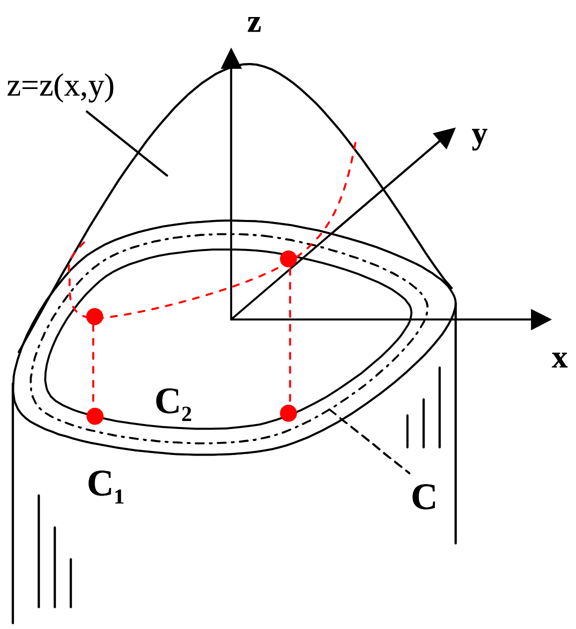
$$\frac{M_z^1}{K_1} = \frac{M_z^2}{K_2} = \ldots\ldots\ldots\ldots\ldots\ldots = \frac{M_z^n}{K_n} = \frac{\sum M_z^i}{\sum K_i} = \frac{M_z}{K_{equivalente}}$$

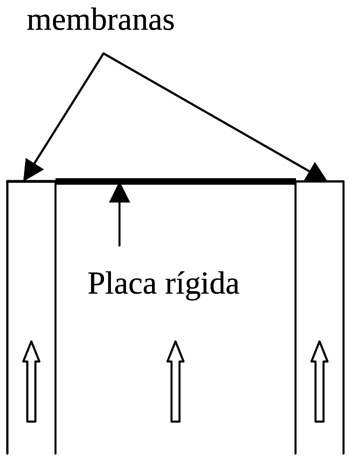
donde: $K_i = \frac{1}{3} e_i^3 l_i$

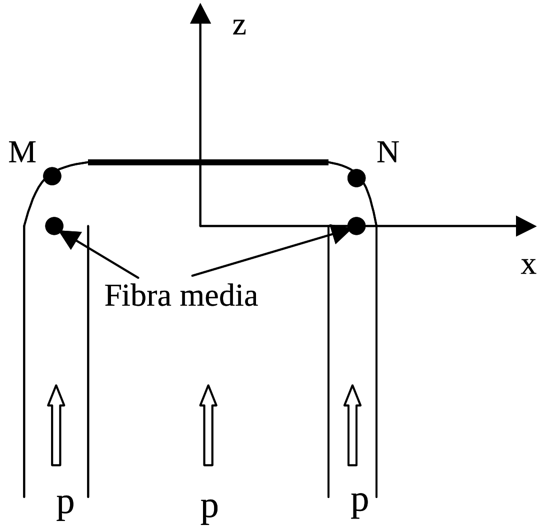
$$M_z^i = \frac{K_i}{K_{equivalente}} M_z$$

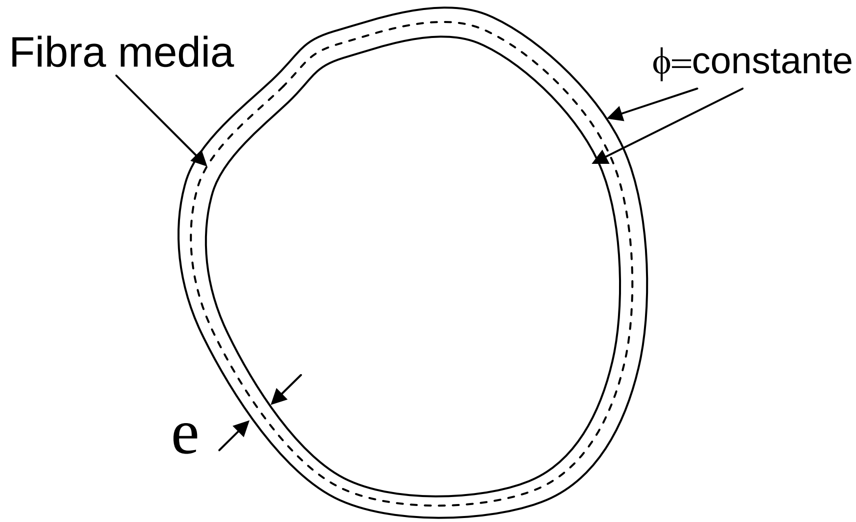
$$\tau_{max} = G\omega e_i = \frac{M_z}{K_{equivalente}} e_i$$

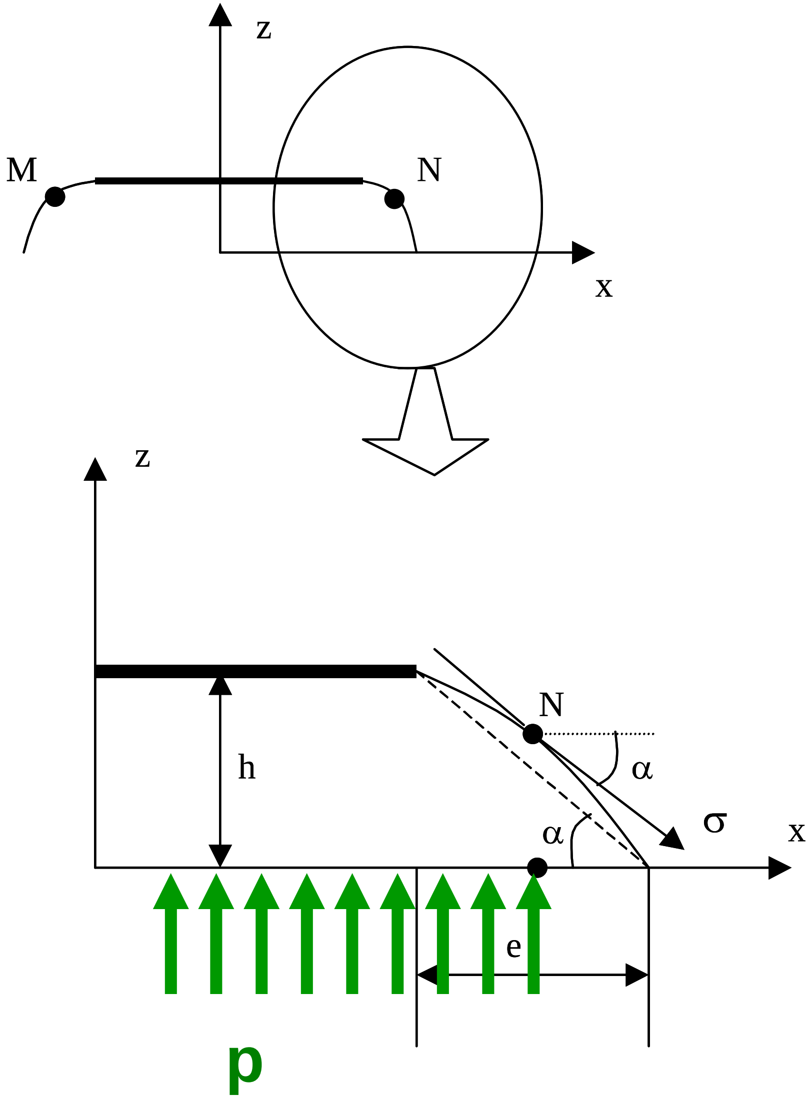
PERFIL CERRADO DE UNA SOLA CELULA



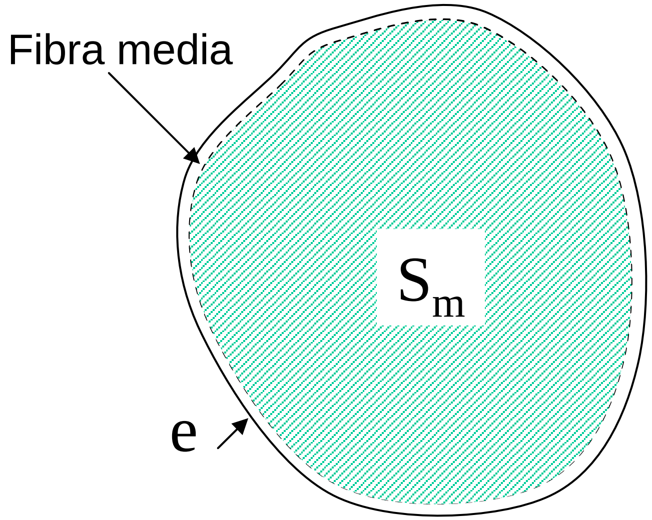

membranas
Placa rígida


z
M
N
x
Fibra media
p
p
p


Fibra media
ϕ=constante
e



Equilibrio de fuerzas según la vertical:



$$pS_m - \oint_C \sigma \, sen\alpha \, ds = 0$$

$$sen\,\alpha = tg\,\alpha = \frac{h}{e}$$

$$pS_m - \oint_C \sigma \frac{h}{e} ds = 0$$

$$pS_m - \sigma \oint_C \frac{h}{e} ds = 0 \quad \Rightarrow \quad \frac{h}{p/\sigma} = \frac{S_m}{\oint_C \frac{ds}{e}}$$

$$\Phi = \frac{2G\omega}{p/\sigma} z \qquad M_z = 2\iint_\Omega \Phi(x,y)\,dx.dy$$

$$M_z = \frac{4G\omega}{p/\sigma}\iint_\Omega z\,d\Omega$$

$$\iint_\Omega z\,d\Omega \cong S_m h$$

$$M_z = \frac{2G\omega}{p/\sigma}(2S_m h) = \frac{4G\omega S_m^2}{\oint_C \frac{ds}{e}}$$

$$\tau_{zy} = \frac{\partial\Phi}{\partial x} = \frac{2G\omega}{p/\sigma}\frac{\partial z}{\partial x} = \frac{2G\omega}{p/\sigma}\frac{h}{e} = \tau_m$$

Cómo:

$$\frac{h}{p/\sigma} = \frac{S_m}{\oint_C \frac{ds}{e}} \qquad \tau_m = \frac{2G\omega}{e}\frac{S_m}{\oint_C \frac{ds}{e}} \qquad \tau_m = \frac{M_z}{2eS_m}$$

$$K = \frac{M_z}{G\omega} = \frac{4S_m^2}{\oint_C \frac{ds}{e}}$$