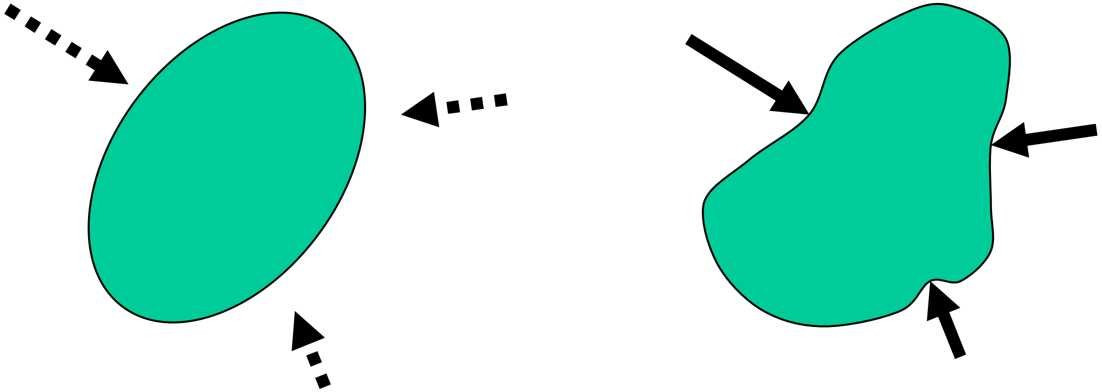
CAPÍTULO 2

DEFORMACIÓN

Al aplicar cargas a un sólido, éste se deforma.

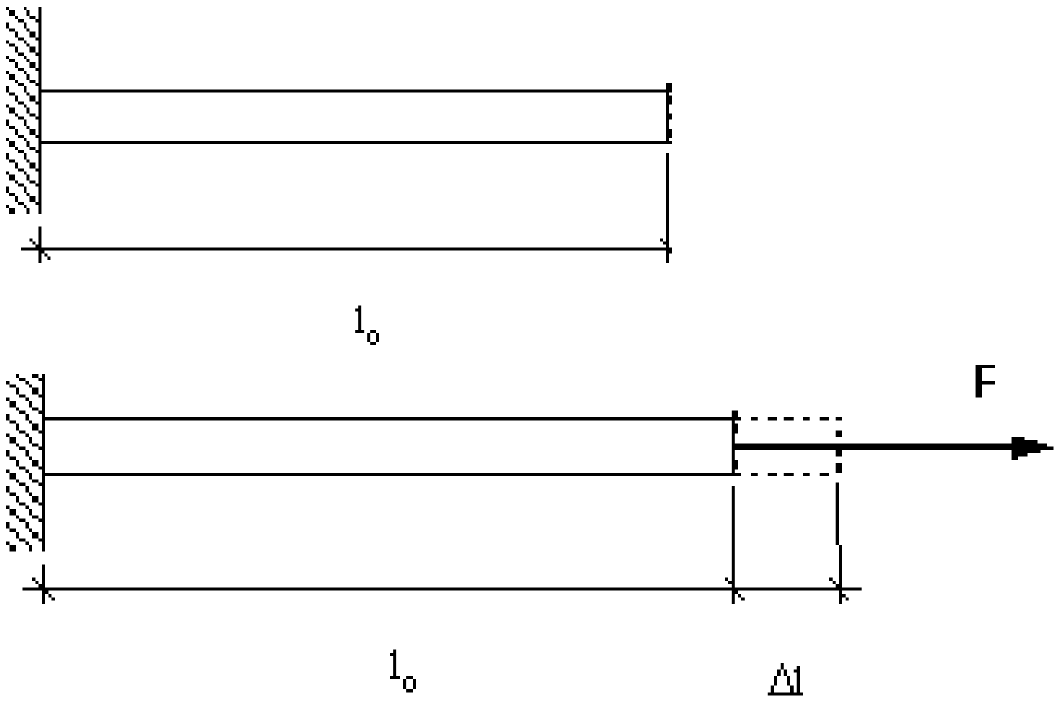


Vamos a suponer que, las deformaciones que se producen dentro del sólido son “pequeñas”de manera tal que, la geometría del sólido antes y después de deformarse es, a efectos prácticos, la misma.

Sólido sin deformar

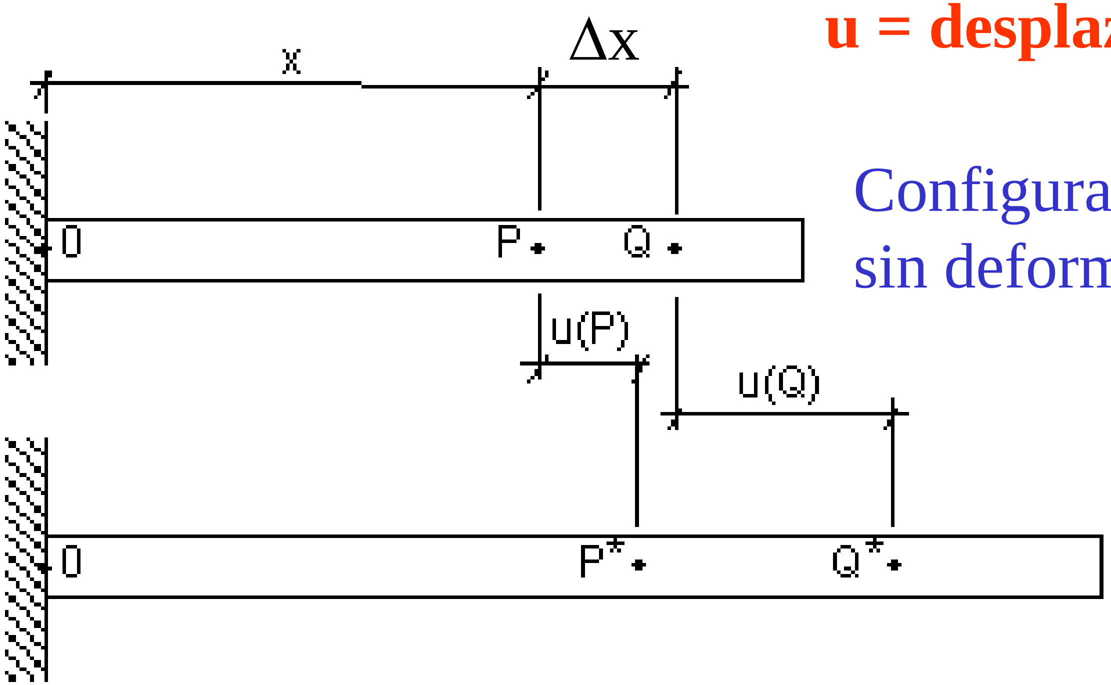
Sólido deformado

DEFORMACION LONGITUDINAL



$$\varepsilon_L = \frac{\Delta l}{l_0}$$

x = posición geométrica
u = desplazamiento experimentado



Configuración sin deformar

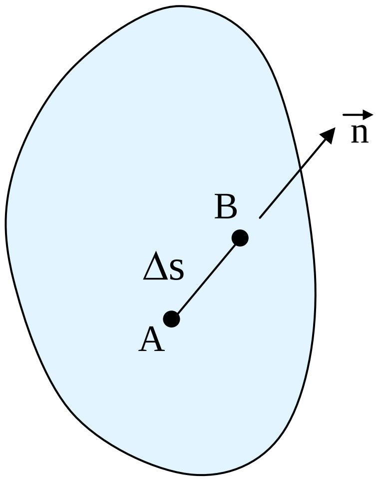
Configuración deformada

$$\varepsilon_x(P) = \lim_{\Delta x \to 0} \frac{P^*Q^* - PQ}{PQ}$$

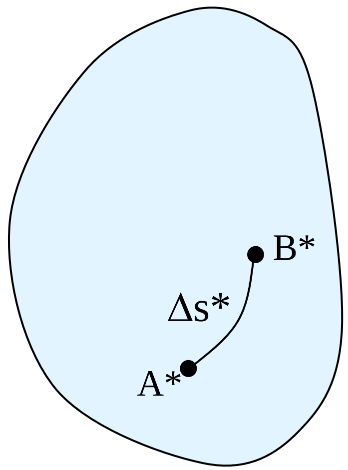
$$P^*Q^* = OQ^* - OP^* = [x + \Delta x + u(Q)] - [x + u(P)]$$

$$P^*Q^* - PQ = u(Q) - u(P) = \Delta u$$

$$\varepsilon_x(P) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \left(\frac{du}{dx}\right)_P$$



Sólido
no deformado



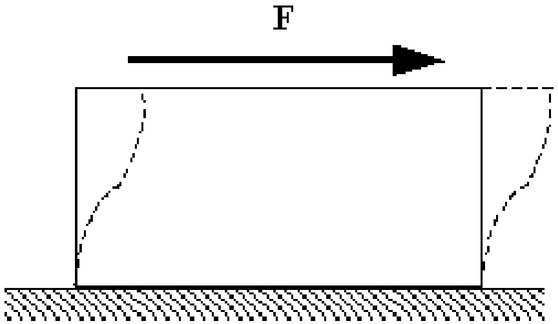
Sólido
deformado

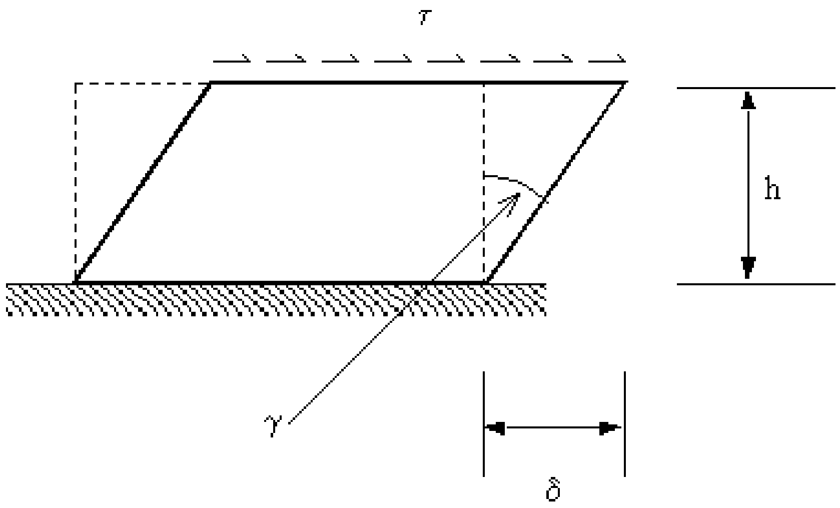
$$\varepsilon = \lim_{B \to A \text{ a a lo largo de n}} \frac{\Delta s^* - \Delta s}{\Delta s}$$

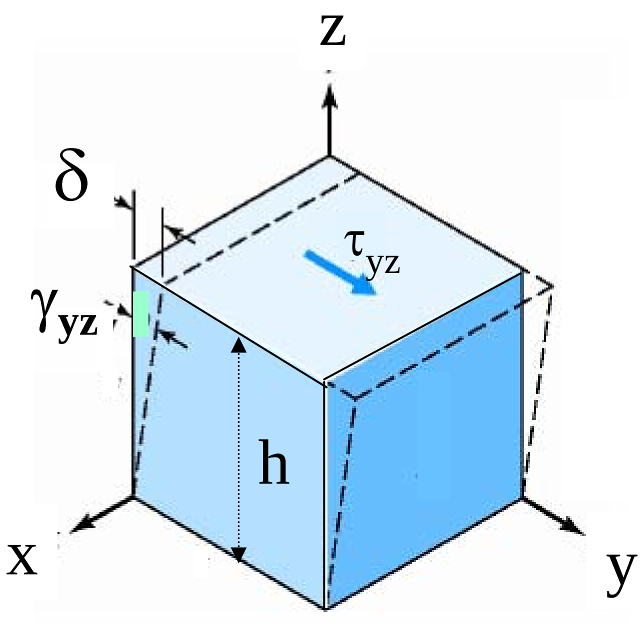
$$\Delta s^* \approx (1+\varepsilon)\Delta s$$

$$\varepsilon \approx \frac{\Delta s^*}{\Delta s} - 1$$

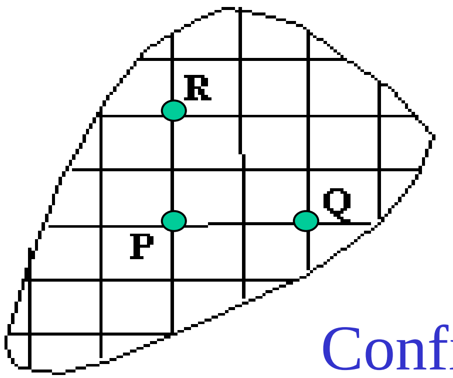
DEFORMACION ANGULAR, TANGENCIAL, DE CORTE O DE CIZALLADURA



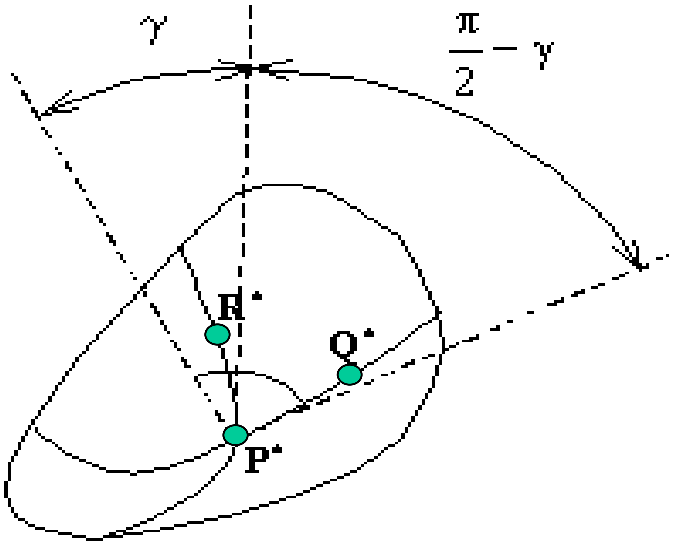




$$\mathbf{tg}\,\gamma_{yz} \approx \gamma_{yz} = \frac{\delta}{h}$$



Configuración
sin deformar

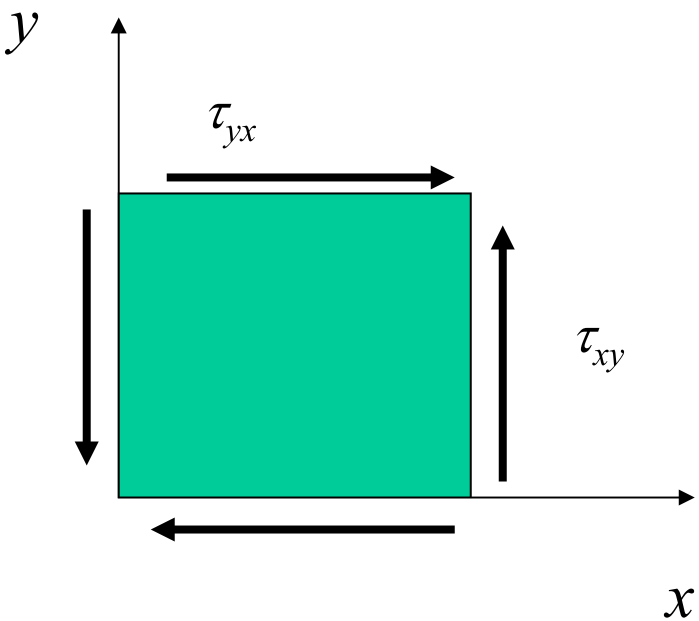


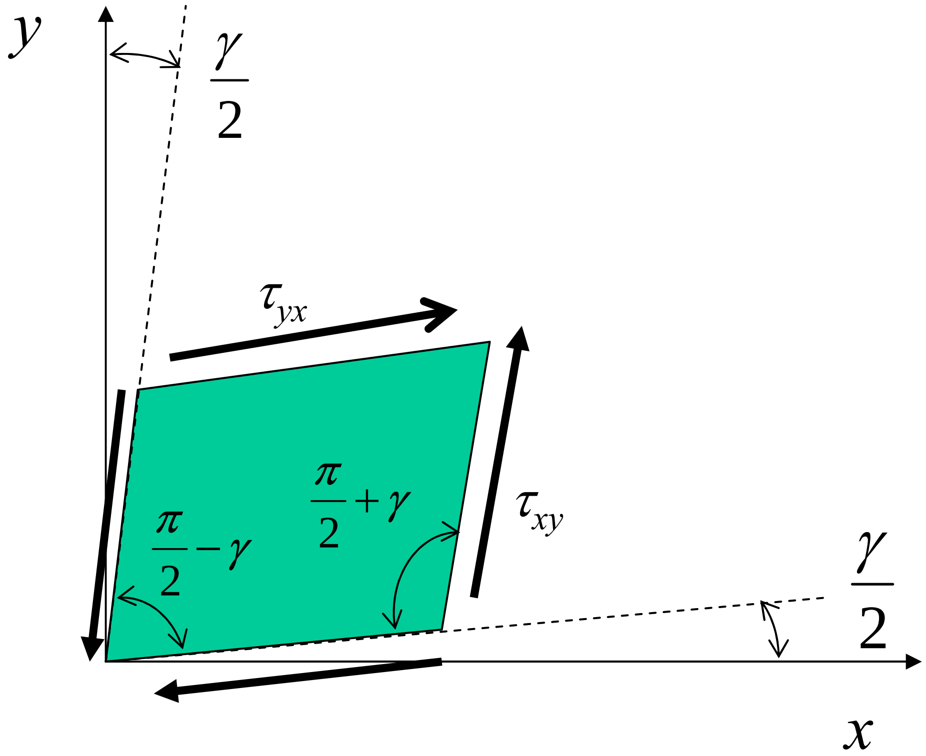
$$\gamma_P = \lim_{\substack{Q\to P\\ R\to P}} \left[\acute{a}ngulo\, QPR - \acute{a}ngulo\, Q^* P^* R^* \right]$$

$$\gamma_P = \lim_{\substack{Q\to P\\ R\to P}} \left[\pi/2 - \acute{a}ngulo\, Q^* P^* R^* \right]$$

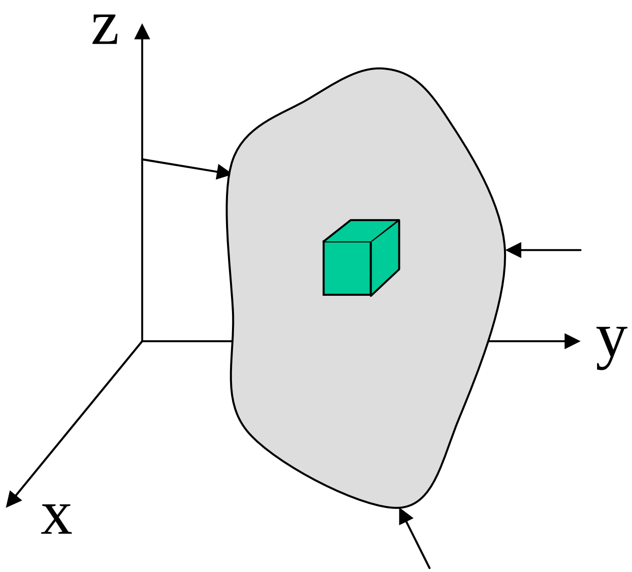
Configuración
deformada

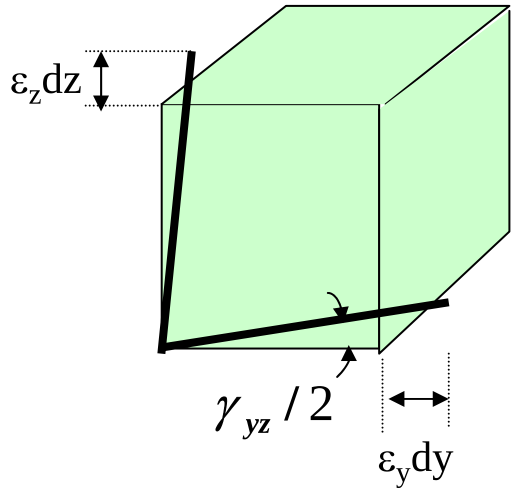
Las tensiones tangenciales actuando en un punto elástico son la causa de aparición de las deformaciones angulares. Estas deformaciones no llevan aparejadas alargamientos o acortamientos del punto elástico sino que, simplemente, distorsionan su geometría.





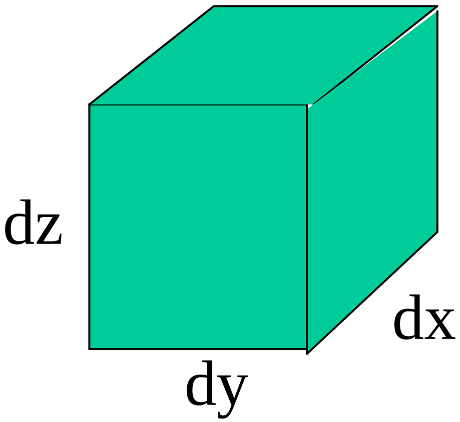
Considerando un punto elástico (dimensiones infinitesimales), podemos determinar sus dimensiones finales así como los ángulos girados por sus lados





Punto elástico deformado

Punto elástico antes de deformarse:



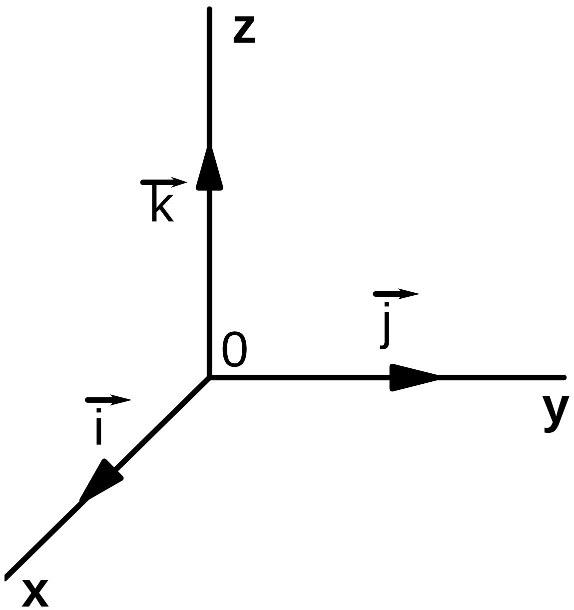
$$(1+\varepsilon_x)dx \qquad (1+\varepsilon_y)dy \qquad (1+\varepsilon_z)dz$$

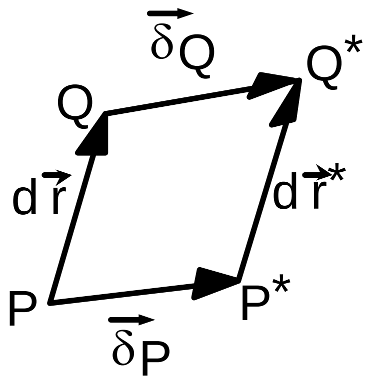
$$\frac{\pi}{2}-\gamma_{xy} \qquad \frac{\pi}{2}-\gamma_{yz} \qquad \frac{\pi}{2}-\gamma_{zx}$$

CAMPO DE DESPLAZAMIENTOS (u,v,w) DENTRO DE UN SÓLIDO

P ⟶ P* Vector desplazamiento en P = $\overrightarrow{PP^*} = \overrightarrow{\delta P}$

Q ⟶ Q* Vector desplazamiento en Q = $\overrightarrow{QQ^*} = \overrightarrow{\delta Q}$





$$\vec{\delta}_P = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\left.\begin{aligned} u&=u(x,y,z)\\ v&=v(x,y,z)\\ w&=w(x,y,z)\end{aligned}\right\}$$ Funciones continuas de x,y,z

$$\vec{\delta}_Q = u'\vec{i} + v'\vec{j} + w'\vec{k}$$

Relación entre (u’,y’,z’) *y* (u,v,w):

$$\left.\begin{aligned}
u' &= u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\
v' &= v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\
w' &= w + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz
\end{aligned}\right\}$$

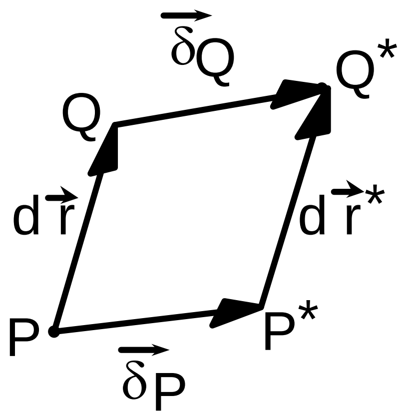
$$\vec{\delta}_Q = \vec{\delta}_P + \left[M\right]d\vec{r}$$

$$\left[M\right] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Descomposición de la matriz [M]

$$\vec{\delta}_Q = \vec{\delta}_P + [M]d\vec{r}$$

$$[M]=\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) & 0 \end{bmatrix}}_{[W]\,hemisimétrica} + \underbrace{\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}}_{[D]\,simétrica}$$



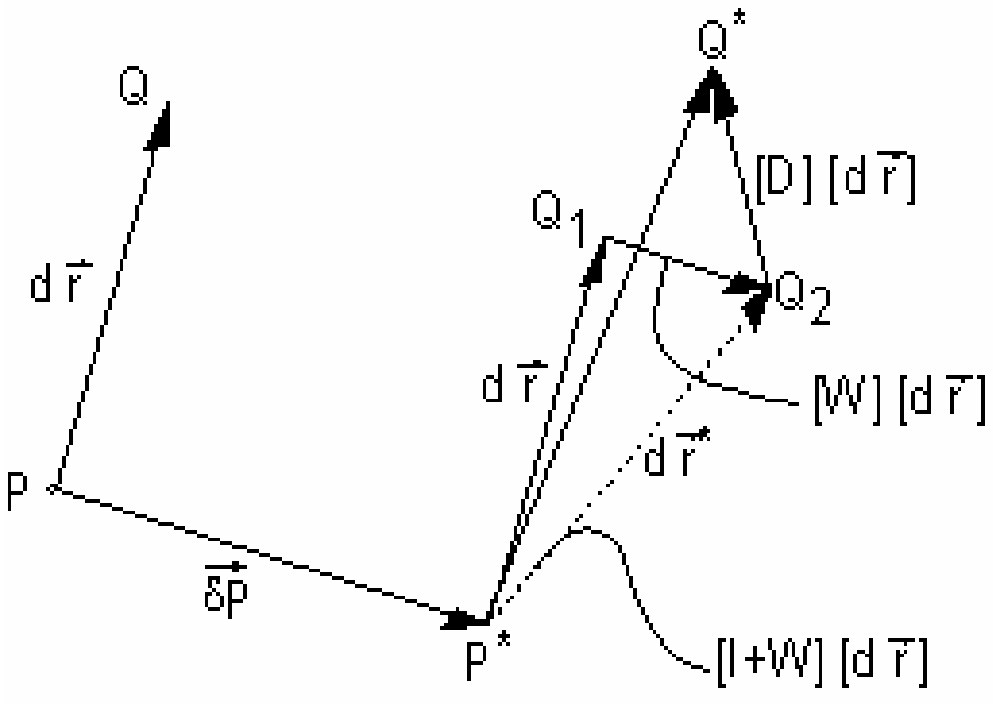
$$\vec{\delta}_Q = \vec{\delta}_P + ([W]+[D])d\vec{r}$$

$$d\vec{r}^* = d\vec{r} + \vec{\delta}_Q - \vec{\delta}_P$$

$$d\vec{r}^* = d\vec{r} + [W]d\vec{r} + [D]d\vec{r}$$

$$d\vec{r}^* = ([I]+[W])d\vec{r} + [D]d\vec{r}$$

Descomposición de movimientos



a) Traslación de definida por $\overrightarrow{PQ} \rightarrow \overrightarrow{P^*Q_1}$

b) Giro definido por la matriz hemisimétrica $\overrightarrow{P^*Q_1} \rightarrow \overrightarrow{P^*Q_2}$

c) Deformación definida por la matriz $\overrightarrow{P^*Q_2} \rightarrow \overrightarrow{P^*Q^*}$

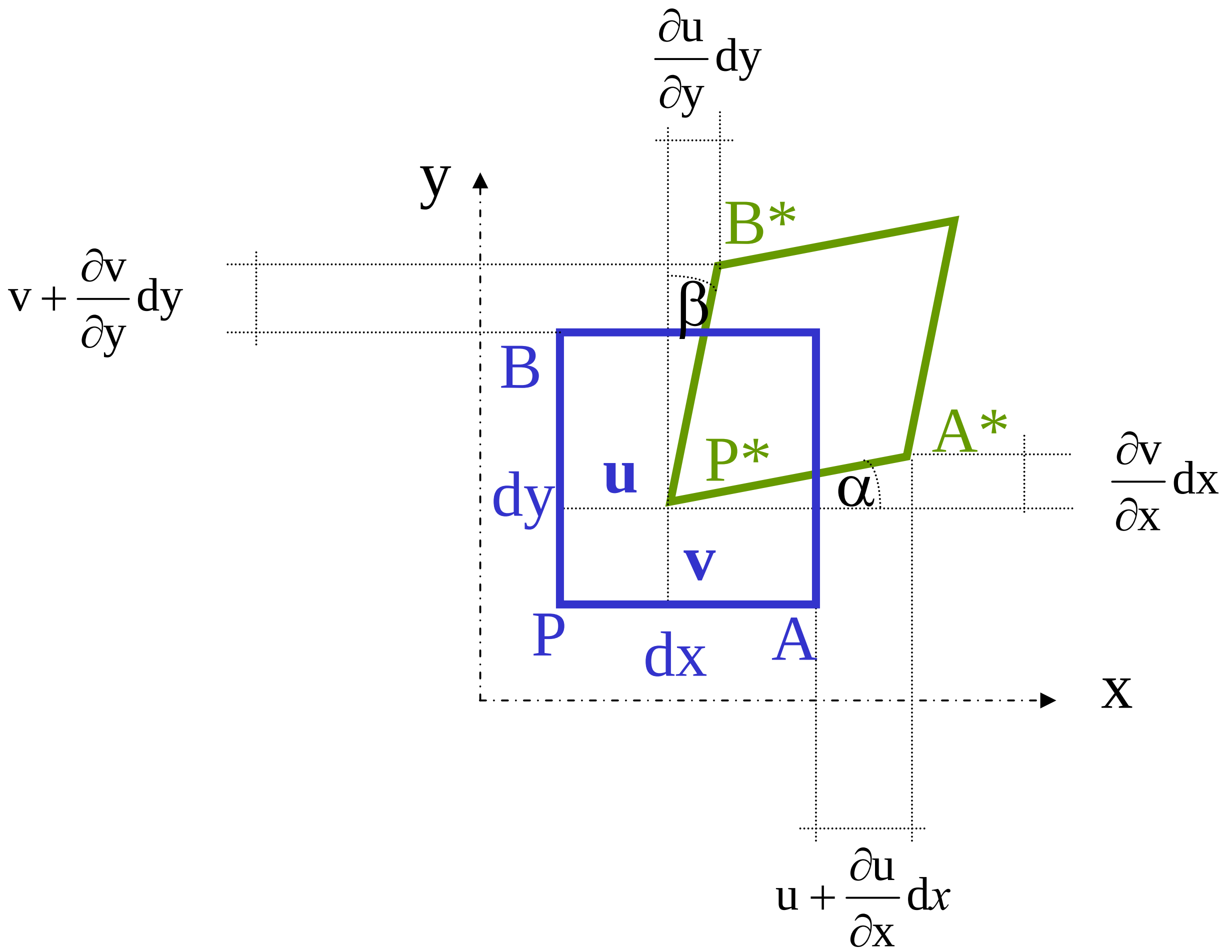
Los pasos ***a)*** y ***b)*** son comunes (traslación + giro) para todos los puntos del entorno del punto P, por lo que no producen variación relativa alguna (deformación) de las distancias entre el punto P y dichos puntos. Sólo el paso ***c)*** es el que produce deformaciones en el entorno del punto P y el tensor correspondiente, que admite una representación a través de la matriz **[D]** respecto al sistema de coordenadas que estamos empleando, se denomina **Tensor de Deformaciones**

INTERPRETACION FISICA DE LAS COMPONENTES DEL TENSOR DE DEFORMACIONES

$$\varepsilon_x = \frac{\partial u}{\partial x} \;,\; \varepsilon_y = \frac{\partial v}{\partial y} \;,\; \varepsilon_z = \frac{\partial w}{\partial z} \;,$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \;,\; \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \;,\; \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

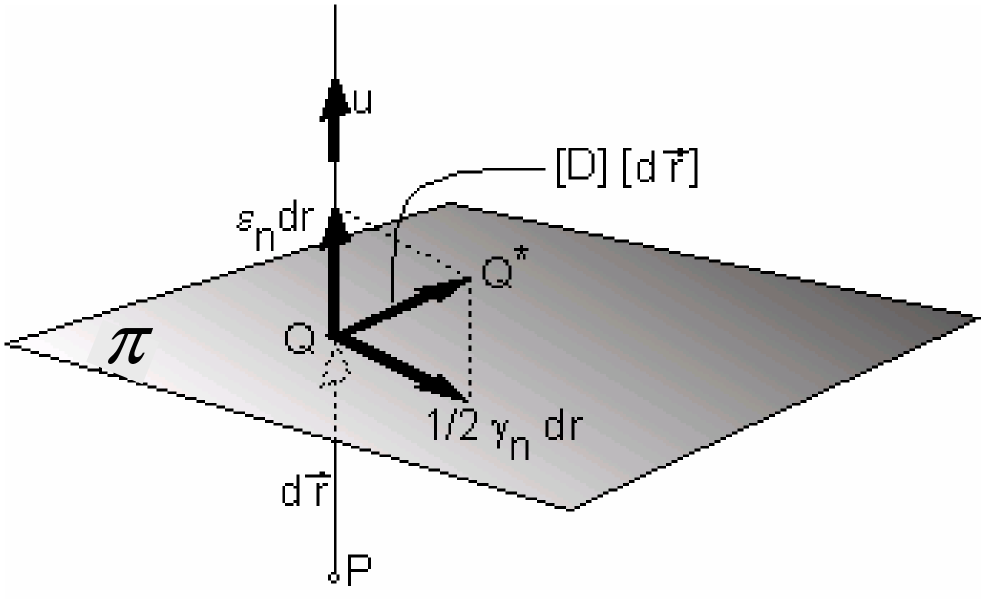
$$[D] = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}$$



$$\left.\begin{aligned} tg\alpha = \alpha = \frac{\partial v}{\partial x} \\ tg\beta = \beta = \frac{\partial u}{\partial y} \end{aligned}\right\} \Rightarrow \gamma_{xy} = \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

DEFORMACIONES EN UNA DIRECCION CUALQUIERA

Vector deformación unitaria: $\vec{\varepsilon}$



$$\vec{\varepsilon} = lim_{\Delta r \to 0} \frac{[D]\Delta\vec{r}}{\Delta r} = [D] lim_{\Delta r \to 0} \frac{\Delta\vec{r}}{\Delta r} = [D]\frac{d\vec{r}}{dr} = [D]\,\vec{u}$$

Componentes intrínsecas de $\vec{\varepsilon}$:

$$\vec{\varepsilon} = [D]\,\vec{u}$$

Deformación longitudinal unitaria, ε_n, definida como:

$$\varepsilon_n = \text{proy. } \vec{\varepsilon} \text{ sobre } \vec{u} = \vec{\varepsilon} \cdot \vec{u} = \left([D] \cdot \vec{u}\right) \cdot \vec{u}$$

$$\varepsilon_n = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{xz} ln$$

Deformación angular unitaria:

$$\gamma_n / 2$$

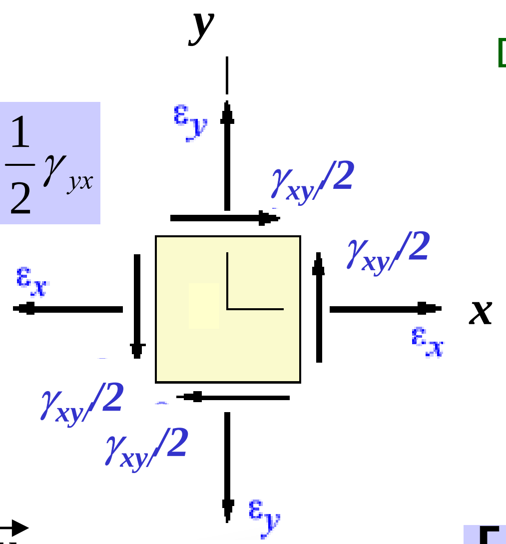
Relación:

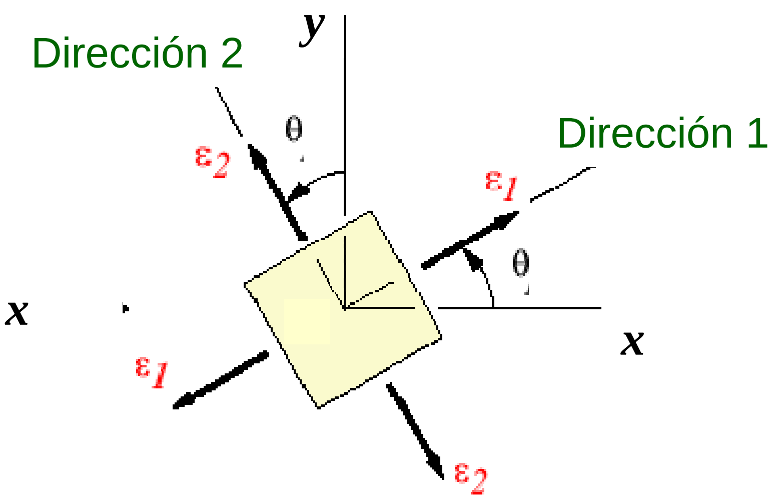
$$\varepsilon^2 = \varepsilon_n^2 + \frac{1}{4}\gamma_n^2$$

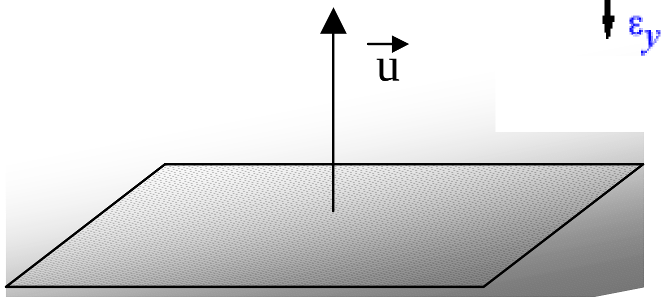
DIRECCIONES PRINCIPALES E INVARIANTES

¿Para qué direcciones el vector deformación es perpendicular al plano correspondiente?

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\gamma_{yx}$$







$$\left[\boldsymbol{D}\right]\vec{\boldsymbol{u}} = \varepsilon\,\vec{\boldsymbol{u}}$$

$$\left[\boldsymbol{D} - \varepsilon \boldsymbol{I}\right]\vec{\boldsymbol{u}} = \vec{0}$$

$$\left|D - \varepsilon\, I\right| = 0$$ ECUACION CARACTERISTICA

$$\Downarrow$$

$$\varepsilon^3 - I_1\varepsilon^2 + I_2\varepsilon - I_3 = 0$$

$$\boxed{\varepsilon^3 - I_1\varepsilon^2 + I_2\varepsilon - I_3 = 0}$$

$$I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$ (Invariante lineal)

$$I_2 = \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{vmatrix} + \begin{vmatrix} \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix} + \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix}$$

(Invariante cuadrático)

$$I_3 = \left| D \right|$$ (Invariante cúbico)

TENSOR DE DEFORMACIONES EXPRESADO EN EJES PRINCIPALES

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

Invariantes:

$$I_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

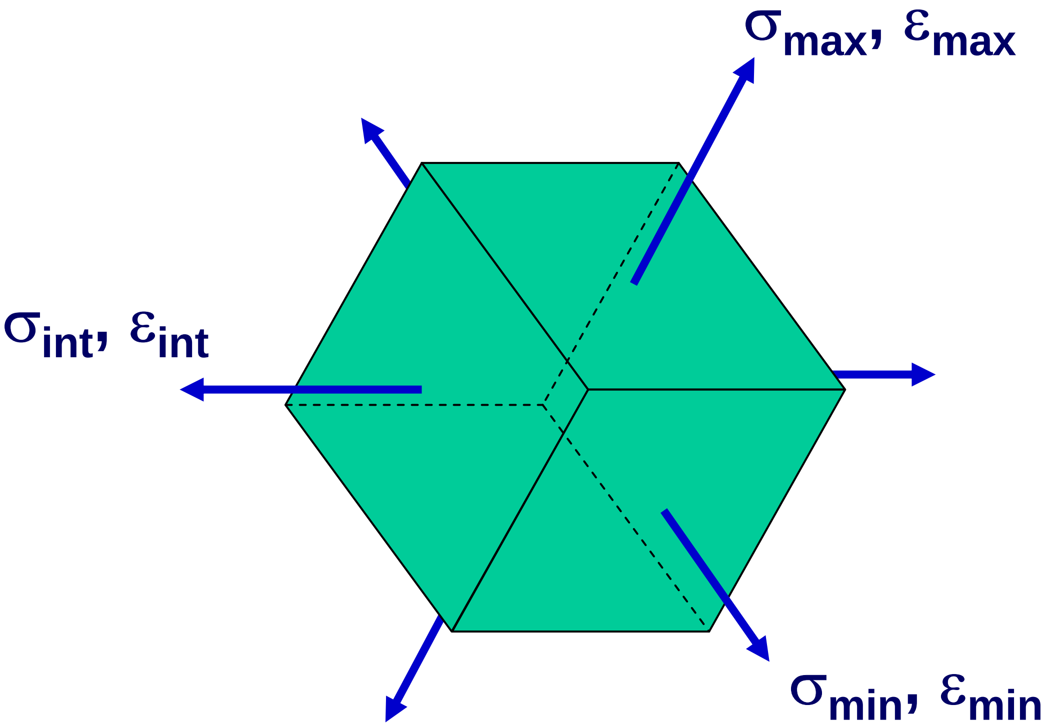
$$I_2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3$$

$$I_3 = \varepsilon_1\varepsilon_2\varepsilon_3$$

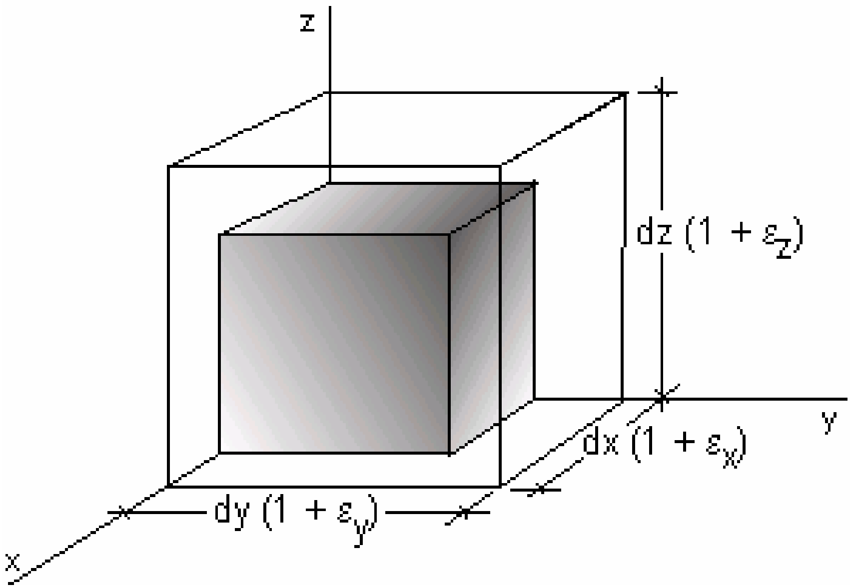
RELACIÓN ENTRE LAS DIRECCIONES PRINCIPALES DE TENSIÓN Y DEFORMACIÓN:

Para un sólido con comportamiento isótropo elástico lineal: $$\gamma = \frac{\tau}{G}$$

Si τ es cero, γ es también nula: Las direcciones principales de tensión Coinciden con las de deformación.



DEFORMACIONES VOLUMETRICA Y DESVIADORA



$$e_V = \frac{Vol.\ final - Vol.inicial}{Vol.inicial}$$

Volumen inicial= dx.dy.dz

Volumen final= $\mathrm{dx} \cdot \mathrm{dy} \cdot \mathrm{dz} \cdot (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z) =$

$$= \mathrm{dx} \cdot \mathrm{dy} \cdot \mathrm{dz} \cdot \left(1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \cancel{[\varepsilon_x \varepsilon_y + \ldots\ldots]}\right)$$

$$e_V = \varepsilon_x + \varepsilon_y + \varepsilon_z = I_1$$

$$\underbrace{\begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}}_{Tensor\ de\ deformacion} = \underbrace{\begin{bmatrix} e_V & 0 & 0 \\ 0 & e_V & 0 \\ 0 & 0 & e_V \end{bmatrix}}_{Comp.\ volumetrica} + \underbrace{\begin{bmatrix} \varepsilon'_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon'_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon'_z \end{bmatrix}}_{Comp.\ desviadora}$$

$$e_V = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\varepsilon'_x = \varepsilon_x - e_V \ ; \ \varepsilon'_y = \varepsilon_y - e_V \ ; \ \varepsilon'_z = \varepsilon_z - e_V$$

ECUACIONES DE COMPATIBILIDAD

$$\vec{\delta}(x.y.z) = u(x,y,z)\vec{i} + v(x,y,z)\vec{j} + w(x,y,z)\vec{k}$$

Las tres funciones *u,v,w* (campo de desplazamientos) no pueden expresarse arbitrariamente en función de *x, y, z*, sino que tendrán que verificar unas determinadas relaciones para que los campos de desplazamientos y de deformaciones que experimenta el sólido sean físicamente posibles.

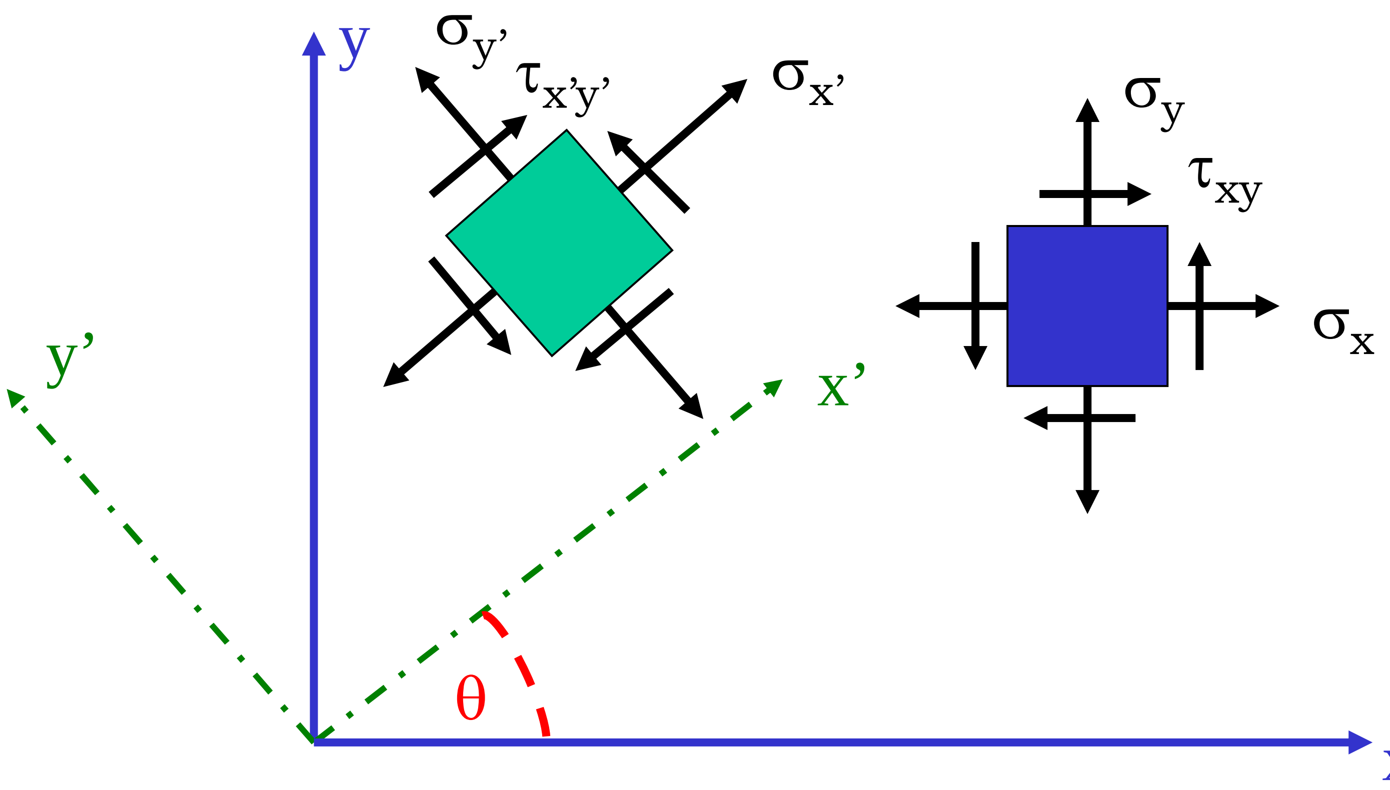
$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y}; \quad 2 \cdot \frac{\partial^2 \varepsilon_x}{\partial y \cdot \partial z} = \frac{\partial}{\partial x} \cdot \left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

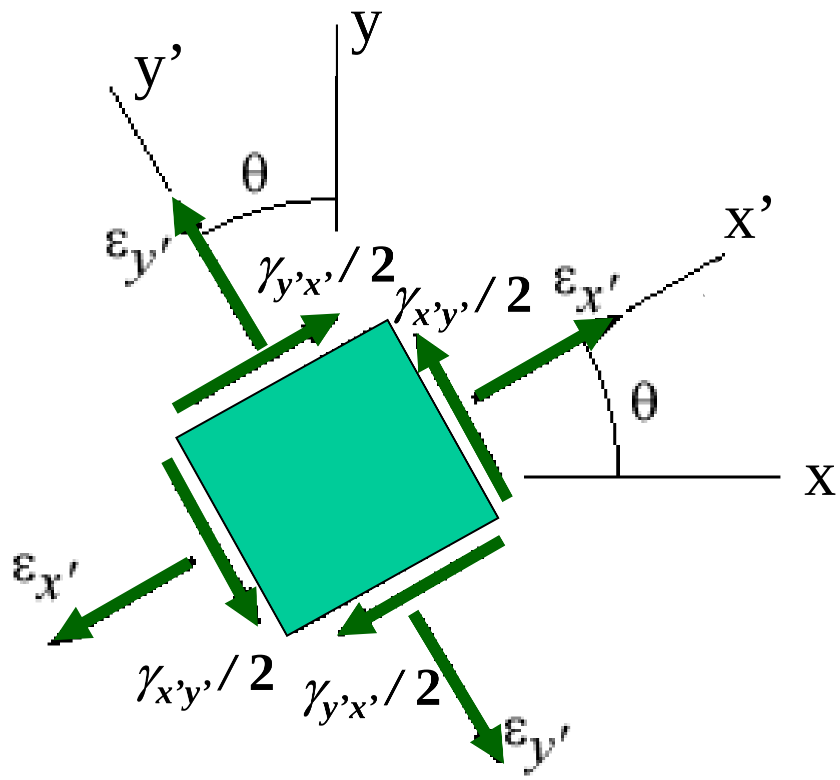
$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \cdot \partial z}; \quad 2 \cdot \frac{\partial^2 \varepsilon_y}{\partial z \cdot \partial x} = \frac{\partial}{\partial y} \cdot \left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \cdot \partial z}; \quad 2 \cdot \frac{\partial^2 \varepsilon_z}{\partial x \cdot \partial y} = \frac{\partial}{\partial z} \cdot \left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right)$$

CAMBIO DEL SISTEMA DE REFERENCIA (2D)

Conocidas las componentes del tensor de deformaciones $(\varepsilon_x, \varepsilon_y, \gamma_{xy}/2)$ en un punto referidas a un sistema cartesiano de referencia *x,y*, veamos cuales son las componentes de dicho tensor respecto de otro sistema cartesiano *x',y'* tal que, el su eje *x'*, forma un ángulo θ. Llamemos $(\varepsilon_{x'}, \varepsilon_{y'}, \gamma_{x'y'}/2)$ a las componentes respecto del nuevo sistema de referencia.



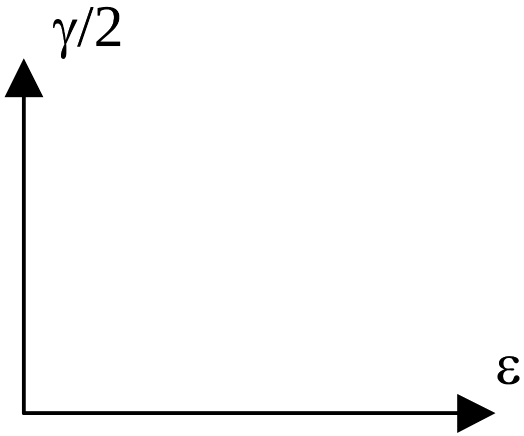


$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\operatorname{sen} 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\operatorname{sen} 2\theta$$

$$\gamma_{x'y'} = -\left(\varepsilon_x - \varepsilon_y\right)\operatorname{sen} 2\theta + \gamma_{xy}\cos 2\theta$$

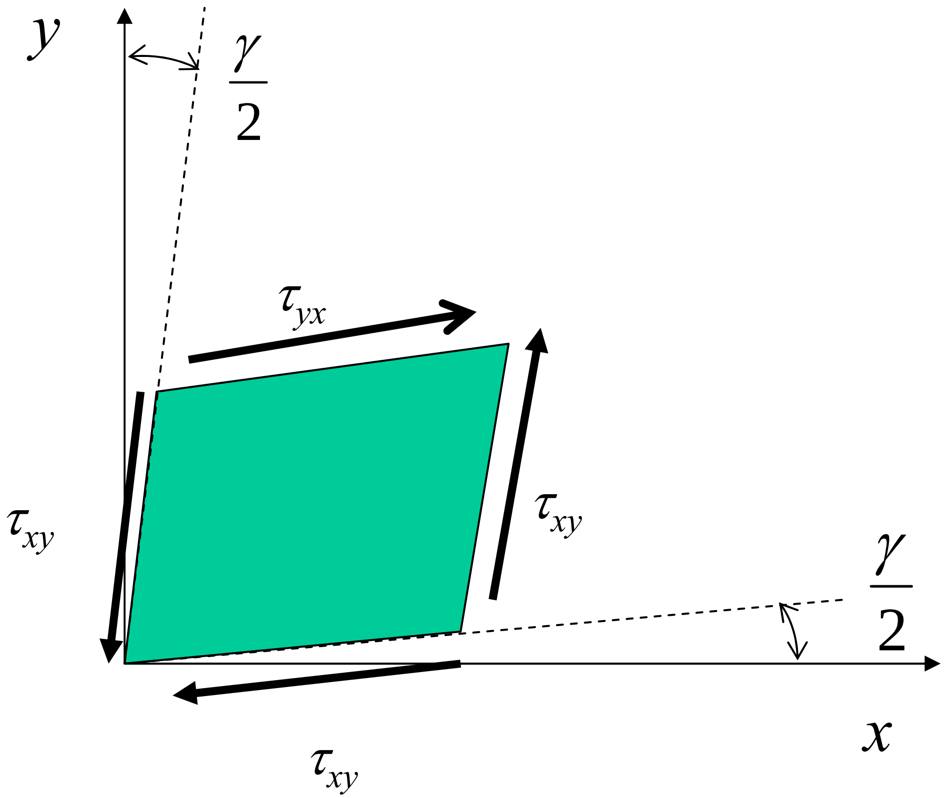
CIRCULO DE MOHR EN DEFORMACIONES

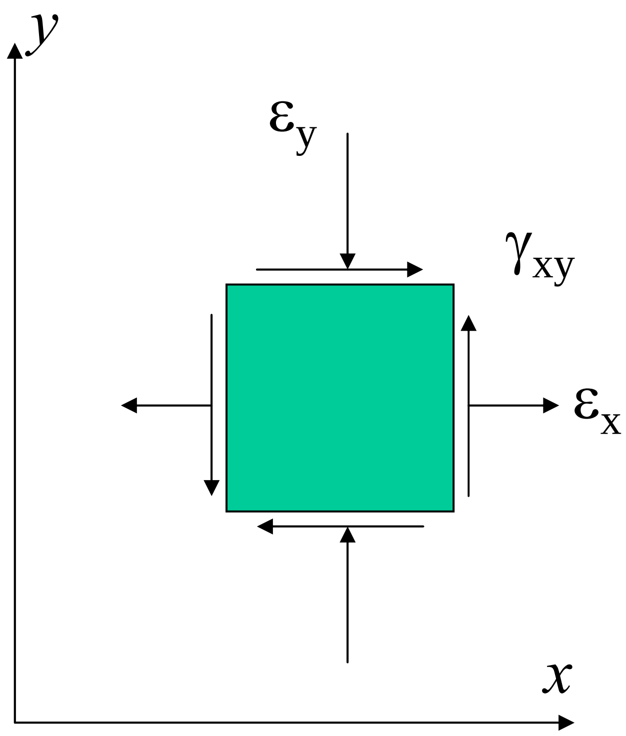


$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \frac{\gamma^2_{x'y'}}{4} = R^2$$

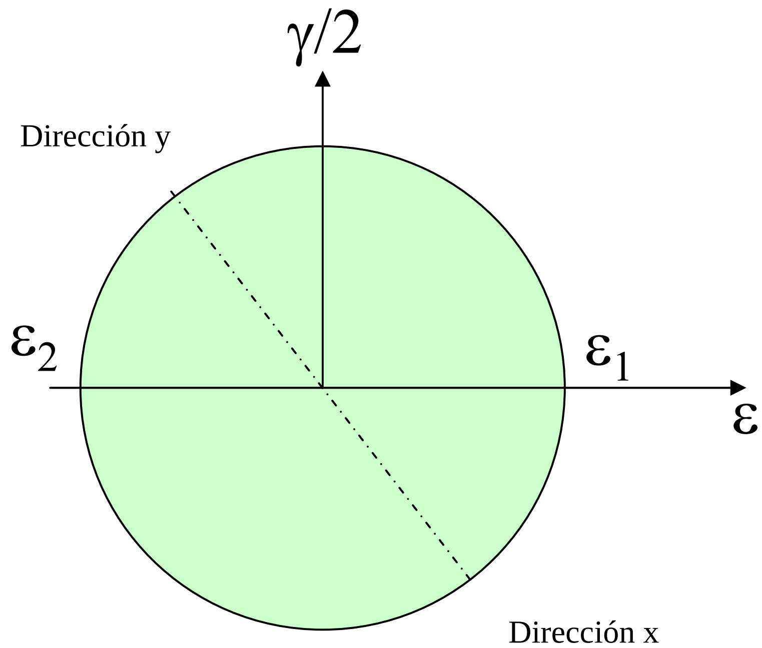
$$R = \sqrt{\left[\frac{\varepsilon_x - \varepsilon_y}{2}\right]^2 + \frac{\gamma^2_{xy}}{4}}$$

CRITERIO DE SIGNOS:





y
εy
γxy
εx
x



γ/2
Dirección y
ε2
ε1
ε
Dirección x