UNIVERSIDAD CARLOS III DE MADRID

PLACAS Y LÁMINAS

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Hipótesis:

- Hipótesis de Kirchhoff:

1.- La rectas perpendiculares al plano medio, antes de que el laminado se deforme, siguen permaneciendo rectas una vez que el laminado se haya deformado.

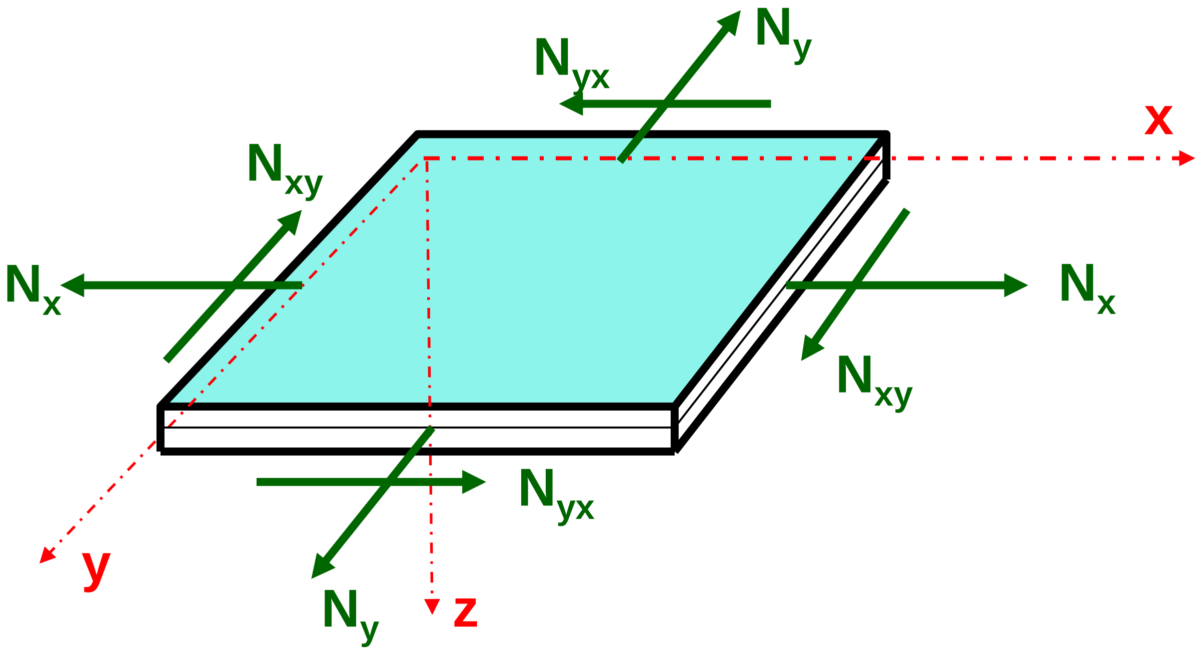
2.- Las rectas perpendiculares al plano medio no experimentan ningún tipo de deformación longitudinal (el laminado no cambia de espesor)

3.- Las rectas perpendiculares al plano medio permanecen perpendiculares a la superficie que adquiere dicho plano una vez que que el laminado flecte.

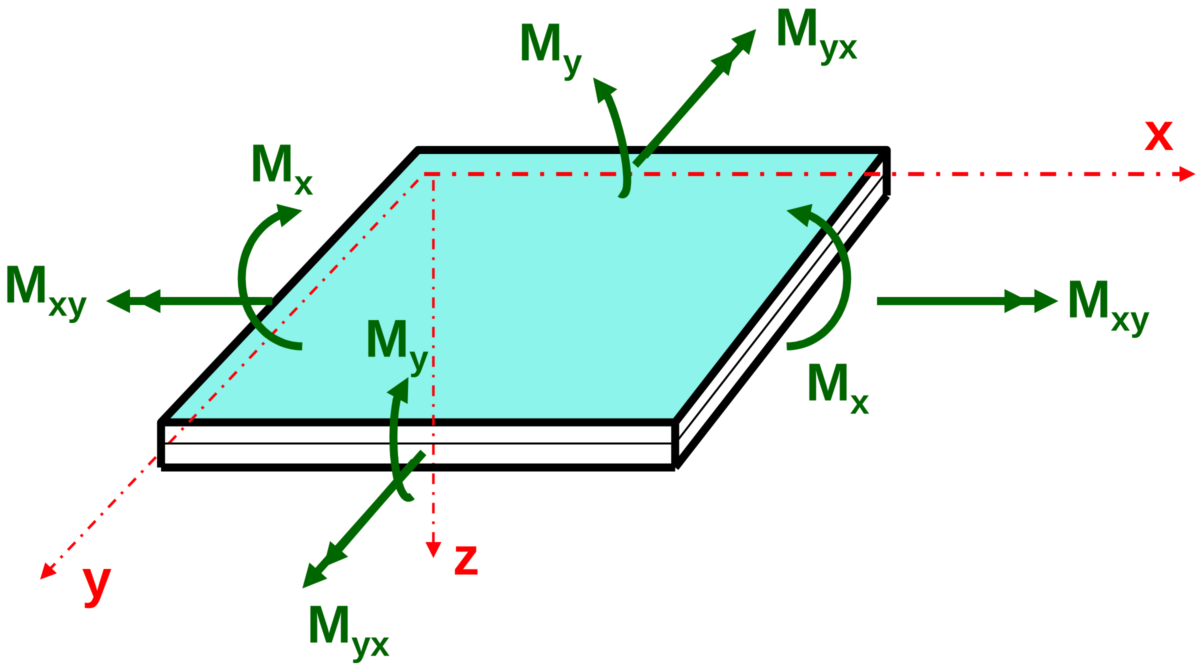
Por tanto, las secciones planas ortogonales al plano medio del laminado siguen siendo planas y ortogonales a la superficie que adquiera dicho plano una vez que el laminado haya flectado.

Hipótesis (Cont.):

- El comportamiento del material se supone elástico lineal.
- Las láminas se encuentran trabajando solidariamente unas a otras
- No existen tensiones fuera del plano de cada lámina ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$): las láminas trabajan en condiciones de tensión plana

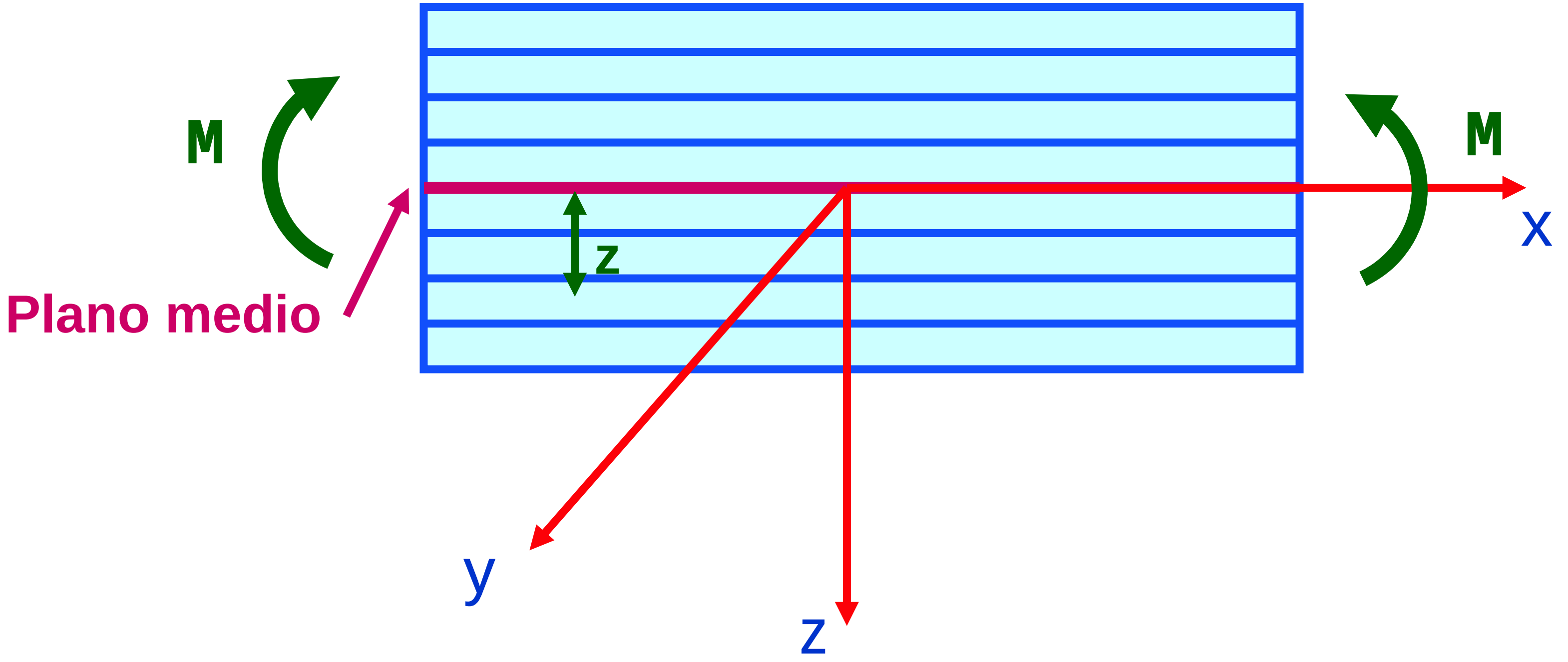


Vector de cargas (N/m): $\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}$

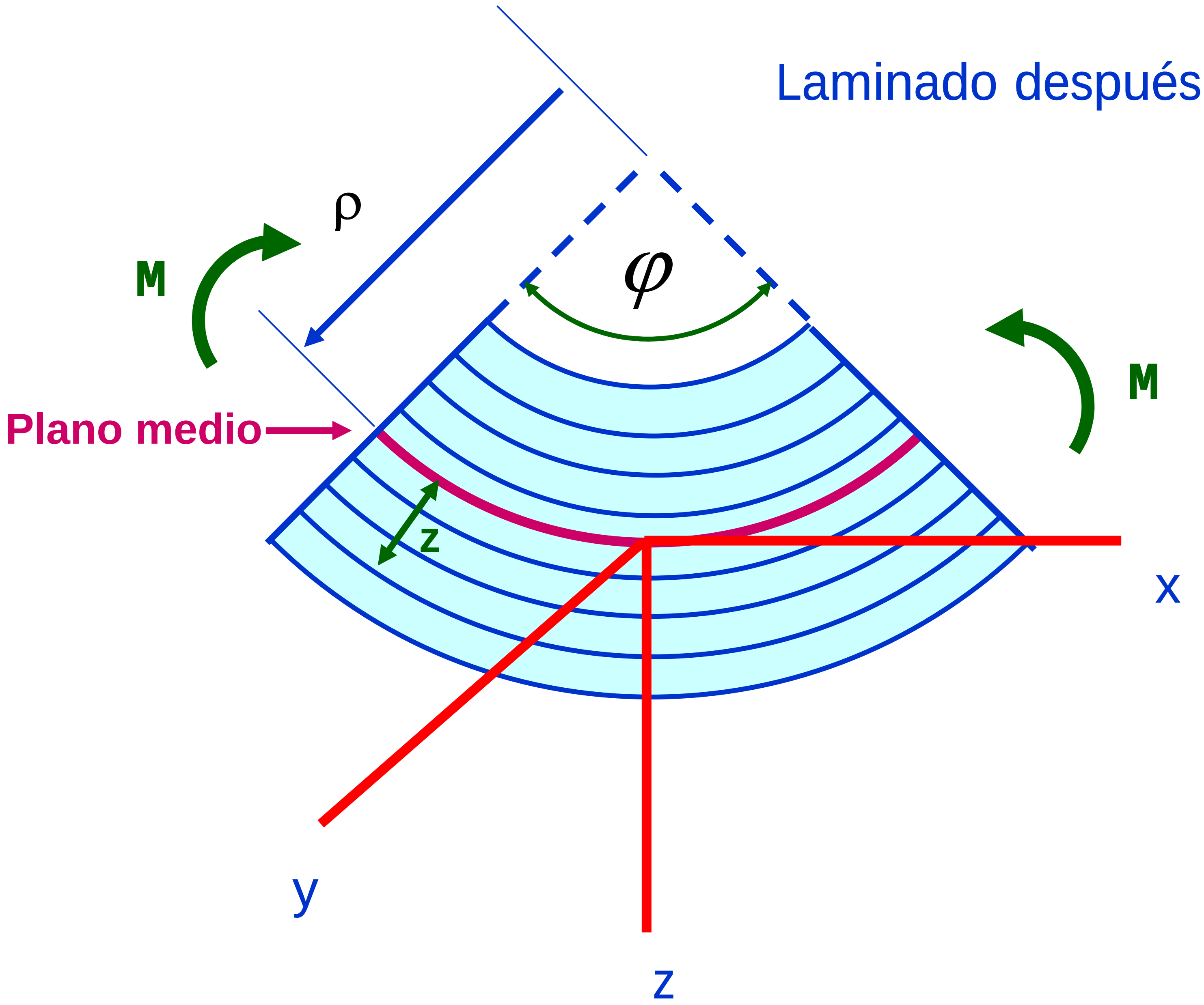


Vector de cargas (Momentos, N.m/m): $\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$

Laminado antes de flectar



Laminado después de flectar



DEFORMACIONES POR FLEXIÓN PURA

$$\varepsilon_x(z) = \frac{(\rho + z)\varphi - \rho\varphi}{\rho\varphi} = \frac{z}{\rho} = z \cdot \kappa$$

donde :

$\rho =$ **radio de curvatura del plano medio durante la flexión**

$\phi =$ **ángulo definido en la figura anterior**

$z =$ **distancia desde el plano medio (definido por el plano xy)**

$\kappa =$ **curvatura**

En definitiva, si se cumplen las tres primeras hipótesis de Kirchhoff:

$$u(x,y,z) = u_0(x,y,z) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y,z) = v_0(x,y,z) - z\frac{\partial w_0}{\partial y}$$

$$w(x,y,z) = w_0(x,y)$$

CAMPO DE DEFORMACIONES EN EL LAMINADO:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_O}{\partial x} - z\frac{\partial^2 w_O}{\partial x^2} \quad \rightarrow -\kappa_x$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_O}{\partial y} - z\frac{\partial^2 w_O}{\partial y^2} \quad \rightarrow -\kappa_y$$

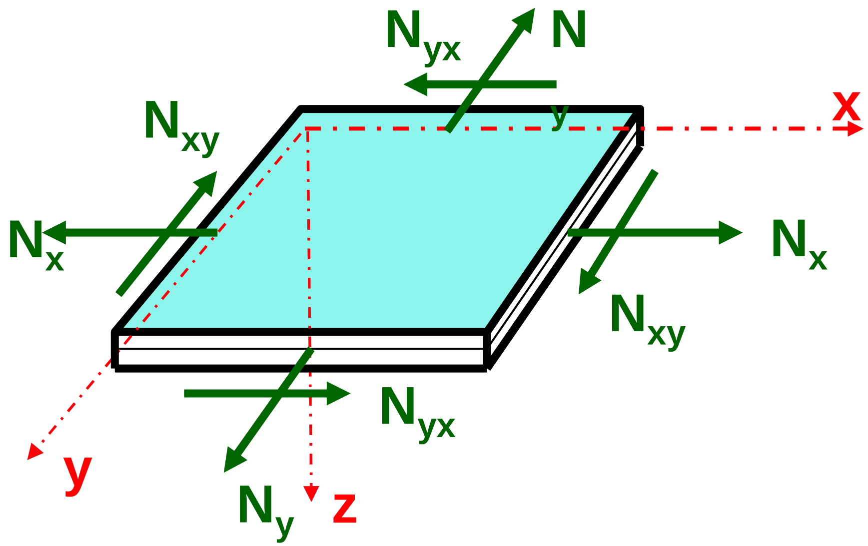
$$\varepsilon_z = 0$$

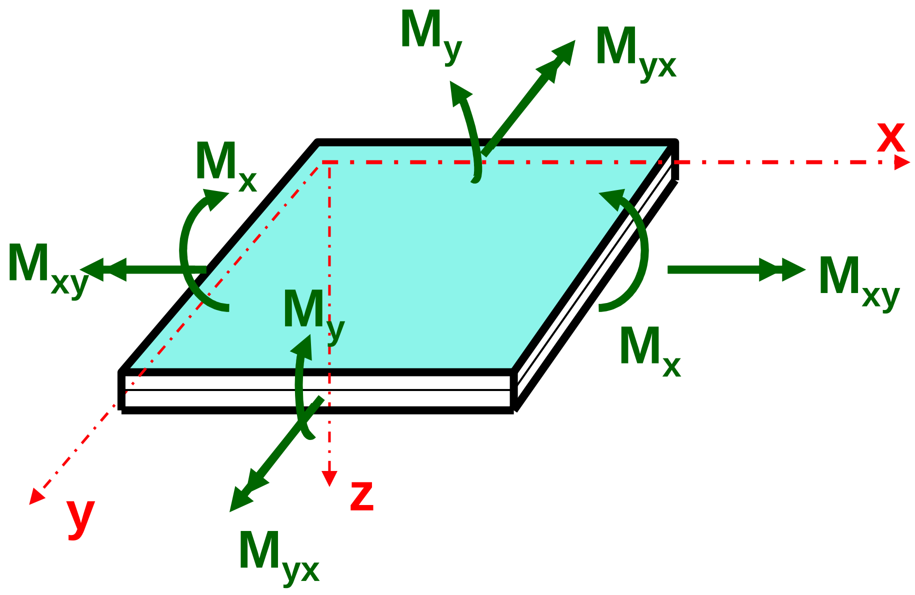
$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} = \frac{\partial u_O}{\partial y} + \frac{\partial v_O}{\partial x} - 2z\frac{\partial^2 w_O}{\partial x \partial y} \quad \rightarrow -\kappa_{xy}$$

$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

TEORÍA CLASICA DE LAMINADOS

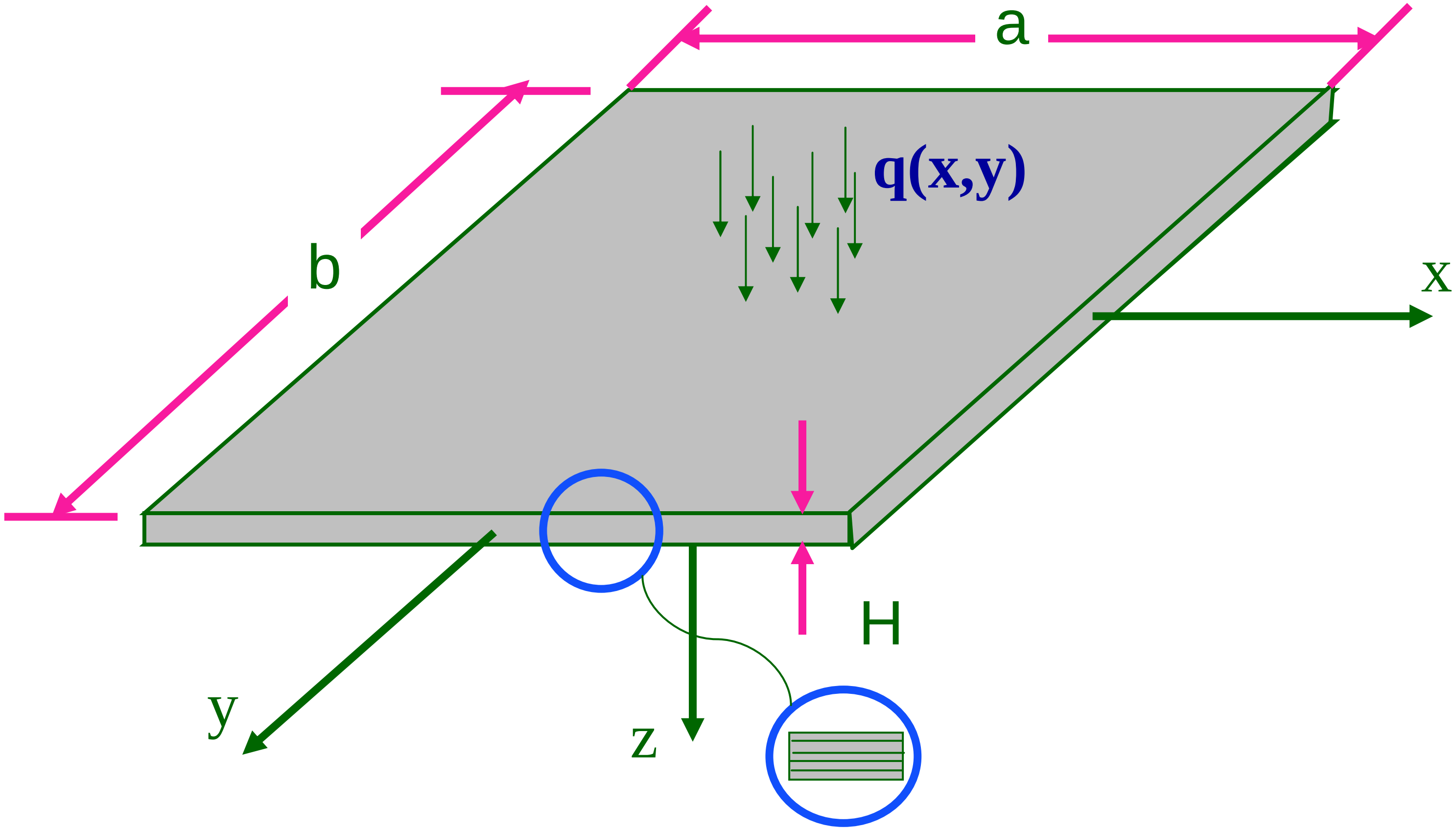


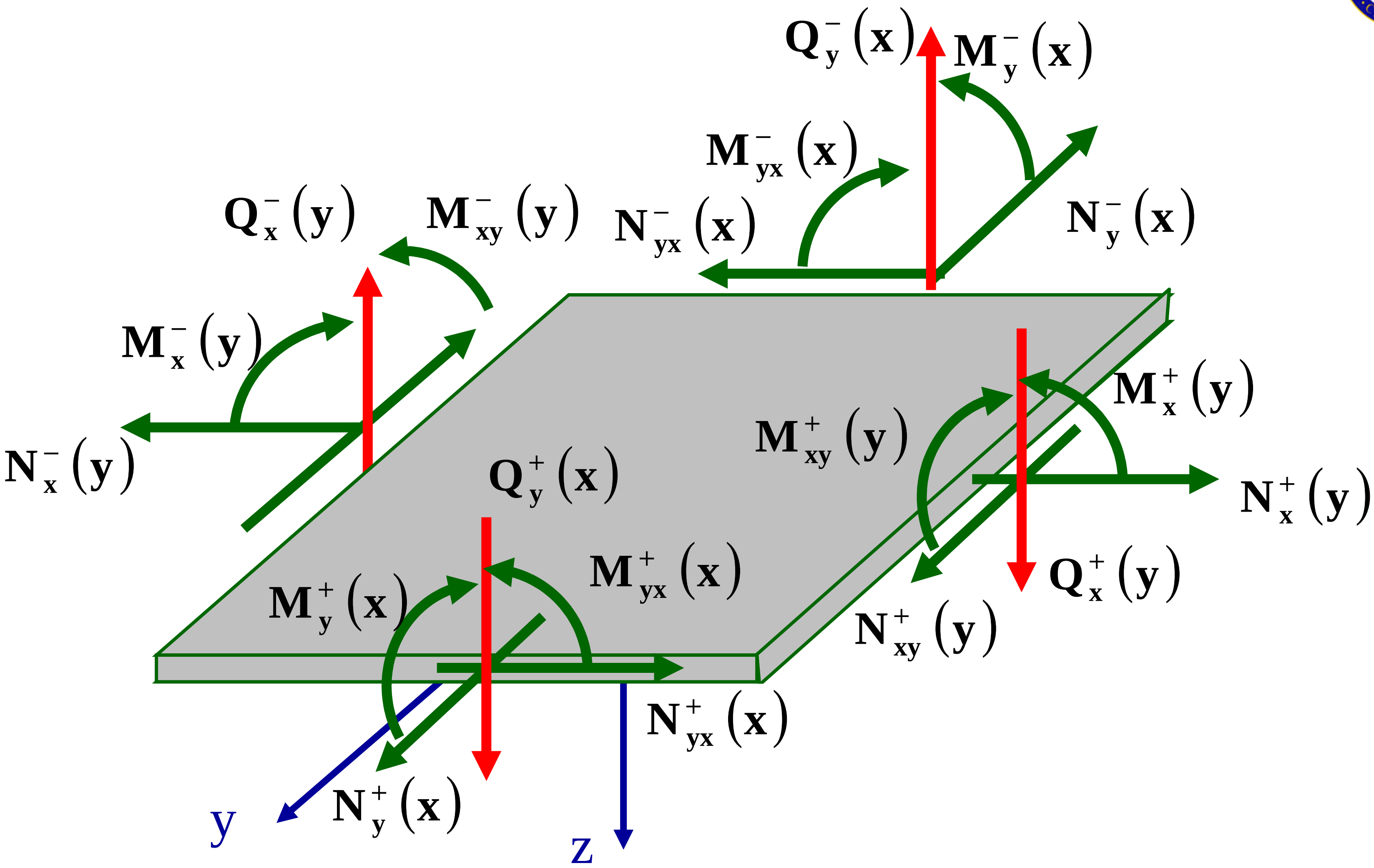


$$\left\{\begin{matrix}\mathbf{N}\\ --\\ \mathbf{M}\end{matrix}\right\} = \left[\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{B} & \mathbf{D}\end{array}\right]\left\{\begin{matrix}\varepsilon^{0}\\ --\\ \kappa\end{matrix}\right\}$$

$$\left\{\begin{matrix}N_x\\ N_y\\ N_{xy}\\ M_x\\ M_y\\ M_{xy}\end{matrix}\right\} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16}\\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26}\\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66}\\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16}\\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26}\\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}\end{bmatrix}\left\{\begin{matrix}\varepsilon^{0}{}_{x}\\ \varepsilon^{0}{}_{y}\\ \gamma^{0}{}_{xy}\\ \kappa_x\\ \kappa_y\\ \kappa_{xy}\end{matrix}\right\}$$

¡En una placa laminada podemos tener cargas fuera del plano!





$Q_y^-(x)$
$M_y^-(x)$
$M_{yx}^-(x)$
$Q_x^-(y)$
$M_{xy}^-(y)$
$N_{yx}^-(x)$
$N_y^-(x)$
$M_x^-(y)$
$N_x^-(y)$
$M_x^+(y)$
$M_{xy}^+(y)$
$Q_y^+(x)$
$N_x^+(y)$
$M_{yx}^+(x)$
$M_y^+(x)$
$Q_x^+(y)$
$N_{xy}^+(y)$
$N_{yx}^+(x)$
y
$N_y^+(x)$
z

OTRAS TEORIAS SOBRE LA FLEXIÓN DE LAMINADOS:

Teoría de primer orden: Se cumplen las dos primeras hipótesis de Kirchhoff pero no la tercera (Las rectas perpendiculares al plano medio **ya no** permanecen perpendiculares a la superficie que adquiere dicho plano una vez que que el laminado flecte)

$$u(x,y,z) = u_0(x,y,z) + z\varphi_x(x,y)$$
$$v(x,y,z) = v_0(x,y,z) + z\varphi_y(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

•**Teoría de segundo orden:** **Se cumple sólo la segunda hipótesis de Kirchhoff**

$$u(x,y,z) = u_0(x,y,z) + z\varphi_x(x,y) + z^2\psi_x(x,y)$$
$$v(x,y,z) = v_0(x,y,z) + z\varphi_x(x,y) + z^2\psi_x(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

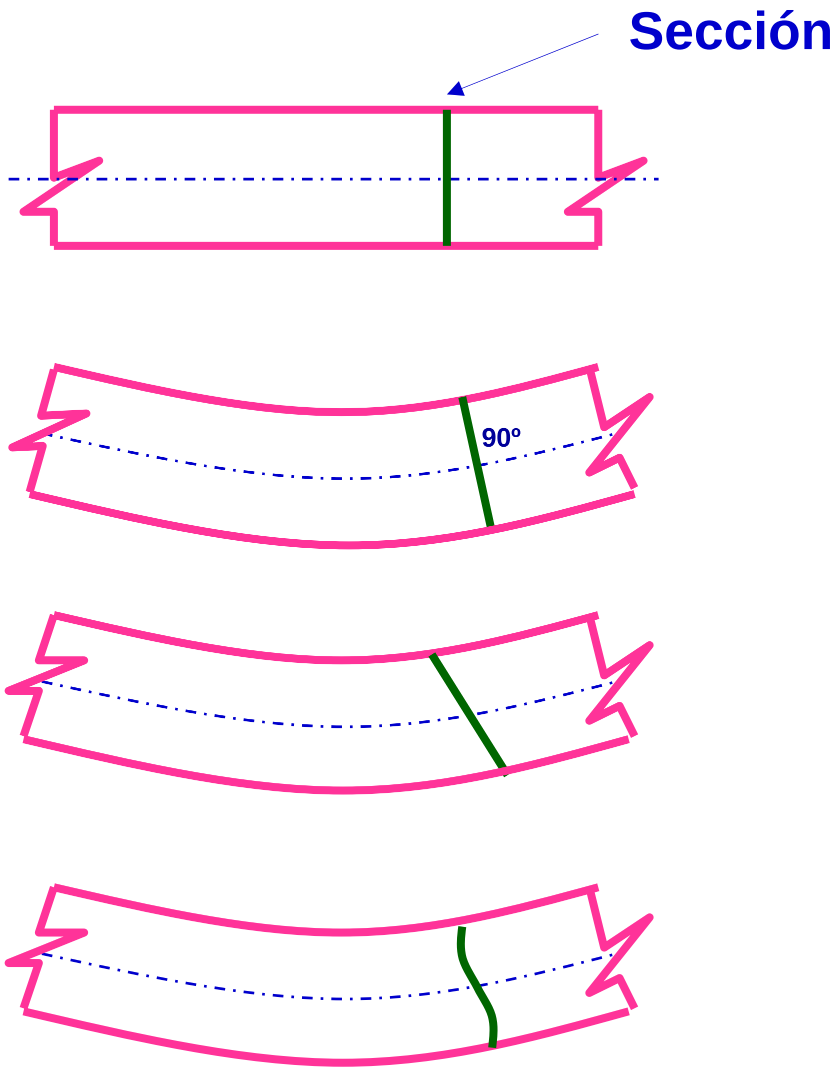
•**Teoría de tercer orden orden (Teoría de Reddy): Se cumple sólo la segunda hipótesis de Kirchhoff**

$$u(x,y,z)=u_0(x,y,z)+z\varphi_x(x,y)+z^3\left(-\frac{4}{3h^2}\right)\left(\varphi_x(x,y)+\frac{\partial w^0}{\partial x}\right)$$

$$v(x,y,z)=v_0(x,y,z)+z\varphi_x(x,y)+z^3\left(-\frac{4}{3h^2}\right)\left(\varphi_y(x,y)+\frac{\partial w^0}{\partial y}\right)$$

$$w(x,y,z)=w_0(x,y)$$

TEORIA DE PLACAS



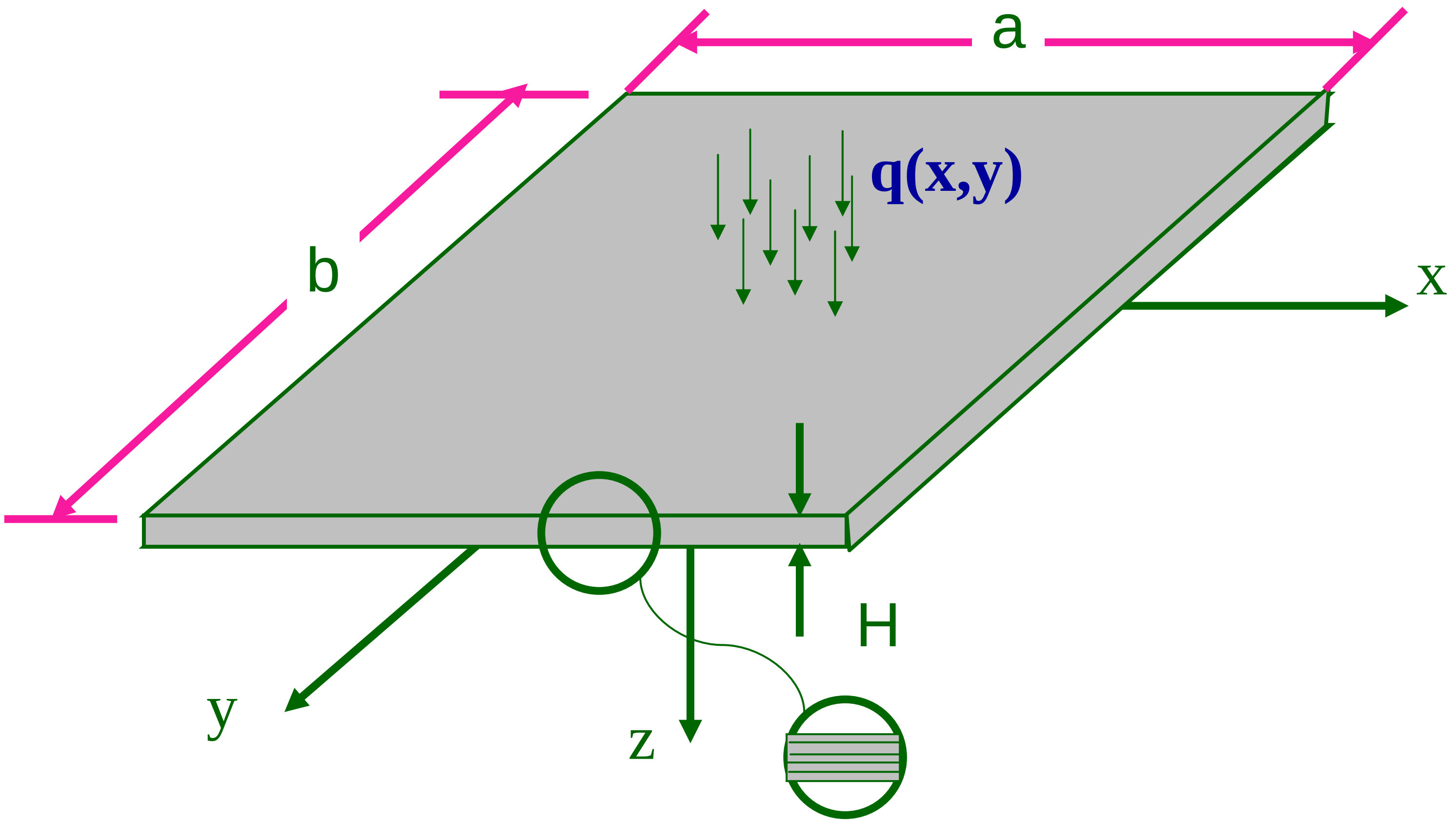
PLACA SIN DEFORMAR

TEORÍA CLÁSICA

TEORÍA DE PRIMER ORDEN

TEORÍA DE TERCER ORDEN

TEORÍA DE PRIMER ORDEN DE PLACAS LAMINADAS



HIPÓTESIS ADICIONALES:

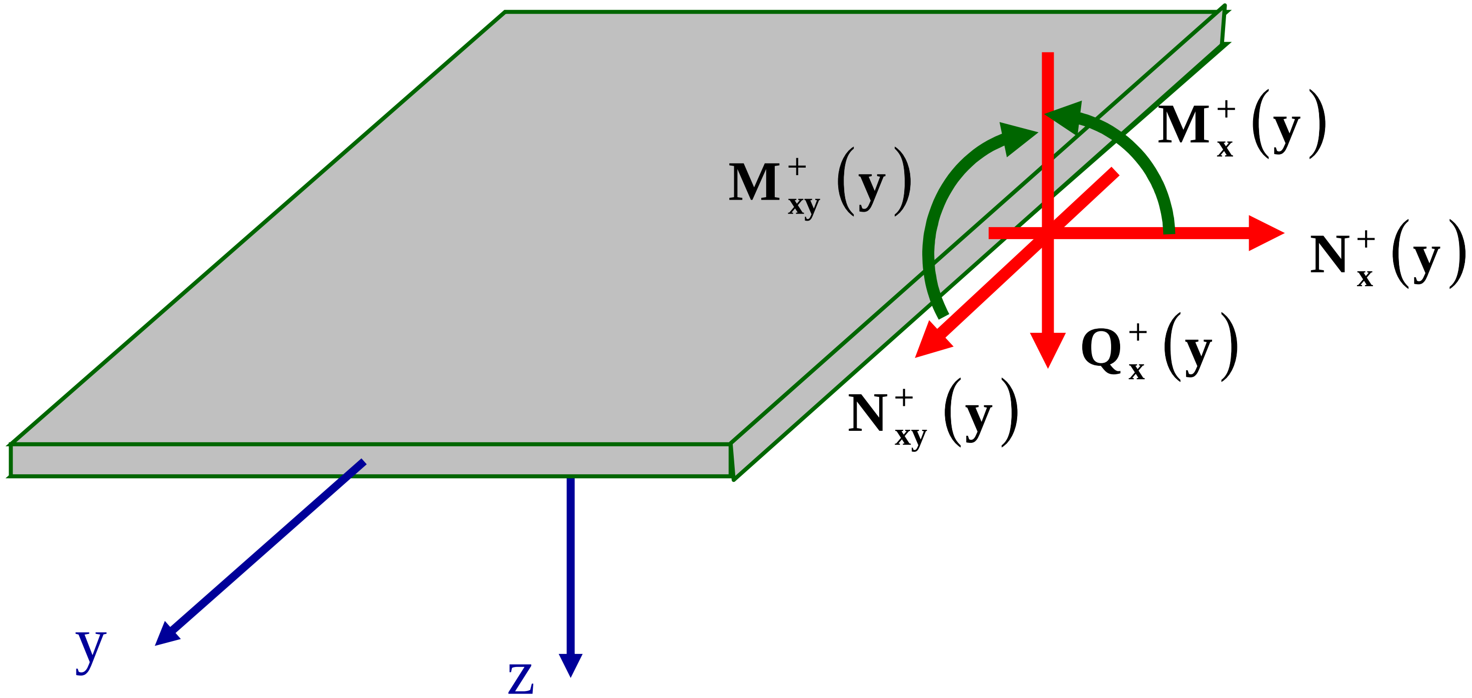
Placa delgada (H << a,b):

$$\sigma_z, \tau_{xz}, \tau_{yz} << \sigma_x, \sigma_y, \tau_{xy}$$

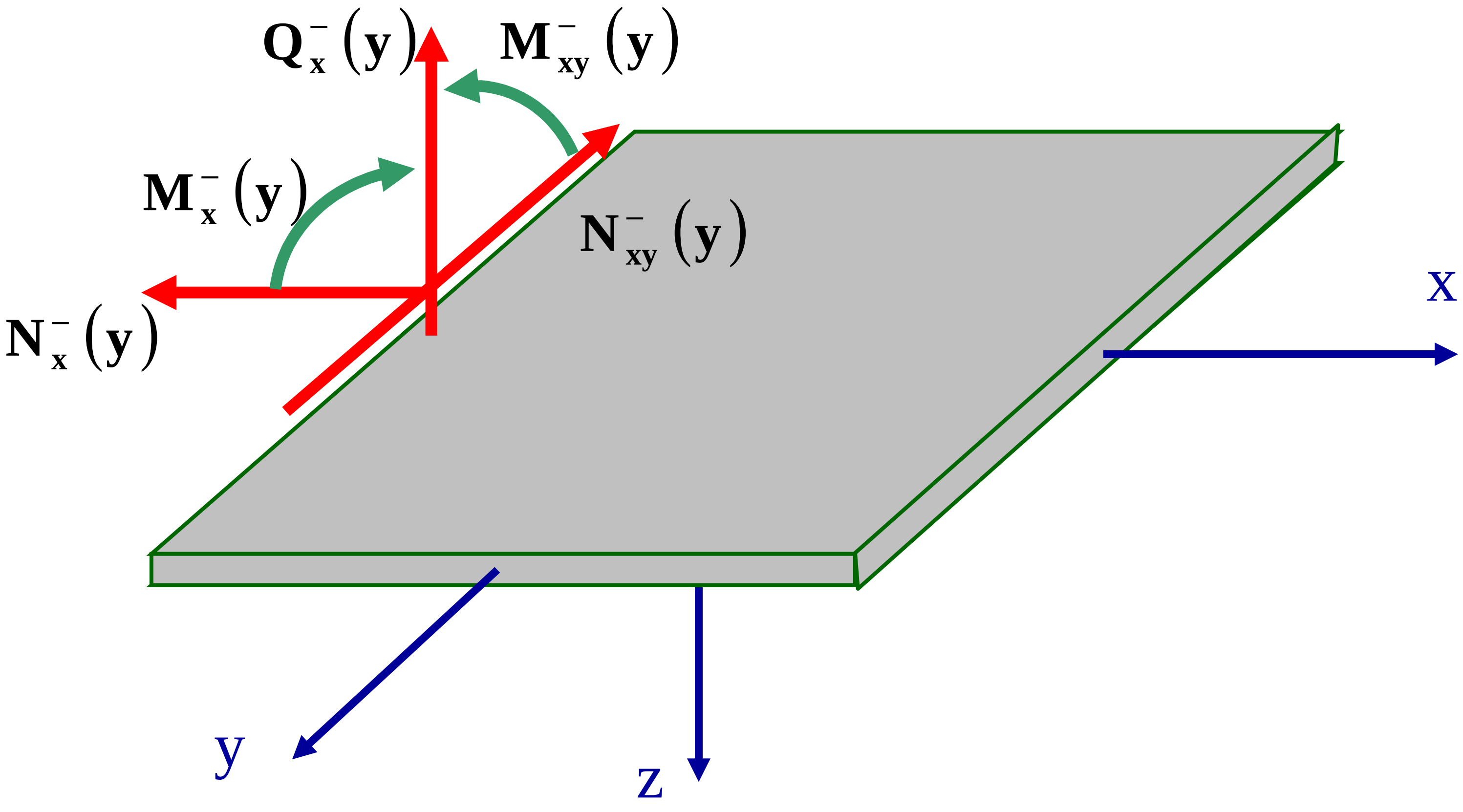
Pequeñas deflexiones ($(w(x,y))_{max}$< H/2):

$$\frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y} << 1$$

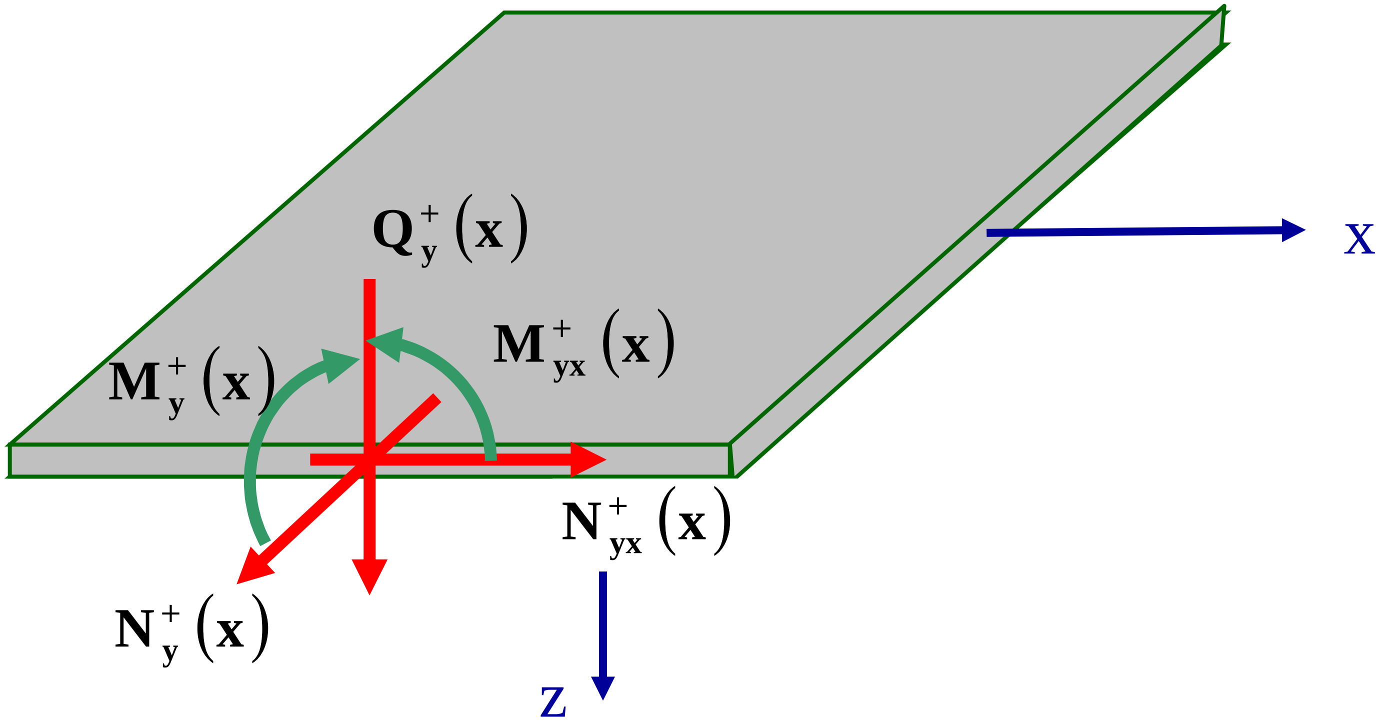
FUERZAS Y MOMENTOS RESULTANTES EN x = +a/2



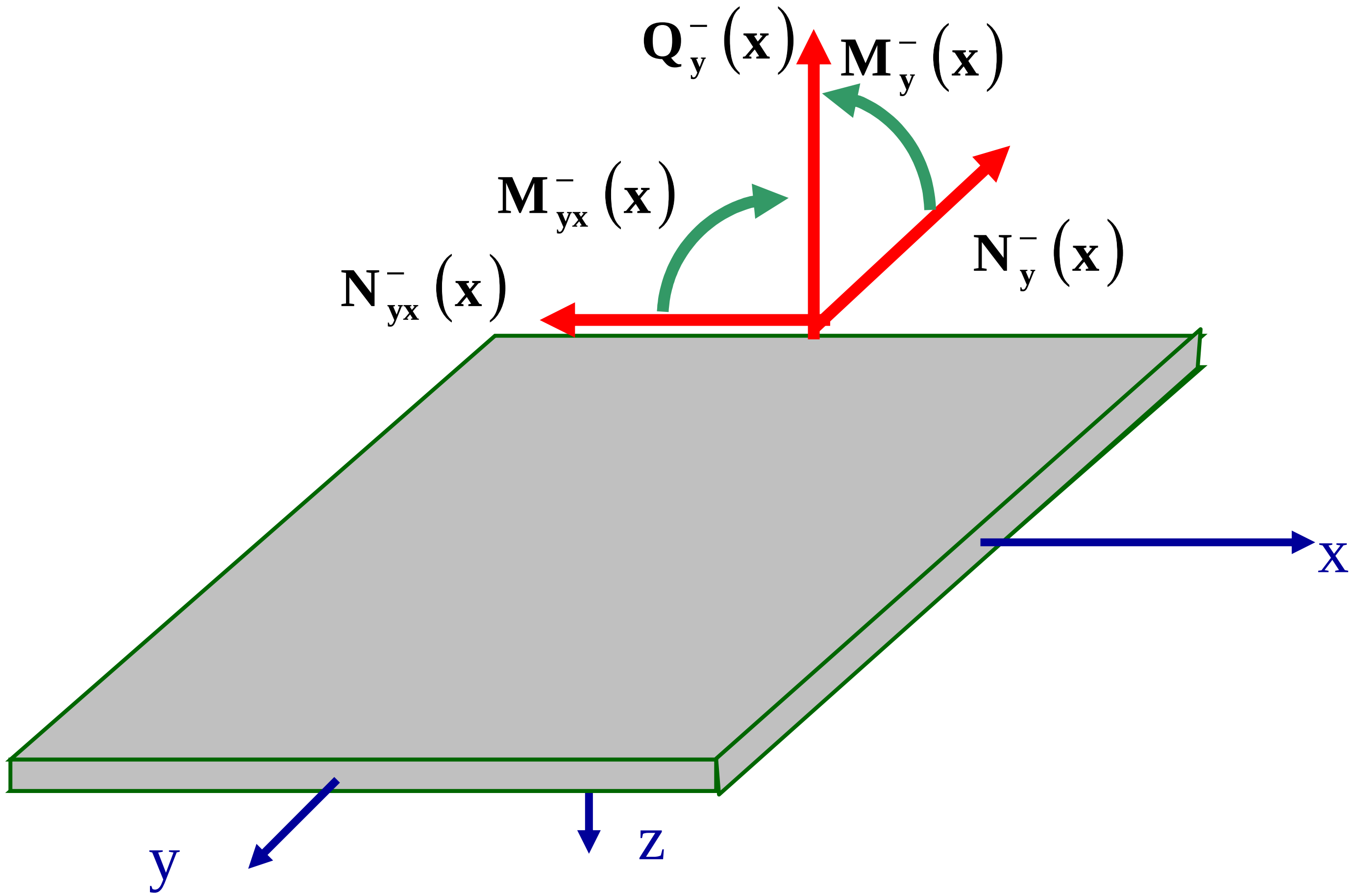
FUERZAS Y MOMENTOS RESULTANTES EN x = -a/2



FUERZAS Y MOMENTOS RESULTANTES EN y= b/2



FUERZAS Y MOMENTOS RESULTANTES EN y= -b/2



RESULTANTES:

Fuerzas normales en las caras:

$$N_x^+(y) = \int_{-H/2}^{H/2} \sigma_x\left(+\frac{a}{2}, y, z\right) dz$$

$$N_x^-(y) = \int_{-H/2}^{H/2} \sigma_x\left(-\frac{a}{2}, y, z\right) dz$$

$$N_y^+(x) = \int_{-H/2}^{H/2} \sigma_y\left(x, +\frac{b}{2}, z\right) dz$$

$$N_y^-(x) = \int_{-H/2}^{H/2} \sigma_y\left(x, -\frac{b}{2}, z\right) dz$$

RESULTANTES:

Fuerzas tangenciales en las caras (en el plano):

$$N_{xy}^{+}(y)=\int_{-H/2}^{H/2}\tau_{xy}\left(+\frac{a}{2},y,z\right)dz$$

$$N_{xy}^{-}(y)=\int_{-H/2}^{H/2}\tau_{xy}\left(-\frac{a}{2},y,z\right)dz$$

$$N_{yx}^{+}(x)=\int_{-H/2}^{H/2}\tau_{yx}\left(x,+\frac{b}{2},z\right)dz$$

$$N_{yx}^{-}(x)=\int_{-H/2}^{H/2}\tau_{yx}\left(x,-\frac{b}{2},z\right)dz$$

MOMENTOS:

Momentos flectores en las caras:

$$M_x^+(y)=\int_{-H/2}^{H/2}\sigma_x\left(+\frac{a}{2},y,z\right)zdz$$

$$M_x^-(y)=\int_{-H/2}^{H/2}\sigma_x\left(-\frac{a}{2},y,z\right)zdz$$

$$M_y^+(x)=\int_{-H/2}^{H/2}\sigma_y\left(x,+\frac{b}{2},z\right)zdz$$

$$M_y^-(x)=\int_{-H/2}^{H/2}\sigma_y\left(x,-\frac{b}{2},z\right)zdz$$

MOMENTOS:

Momentos torsores en las caras:

$$M_{xy}^{+}(y)=\int_{-H/2}^{H/2}\tau_{xy}\left(+\frac{a}{2},y,z\right)zdz$$

$$M_{xy}^{-}(y)=\int_{-H/2}^{H/2}\tau_{xy}\left(-\frac{a}{2},y,z\right)zdz$$

$$M_{yx}^{+}(x)=\int_{-H/2}^{H/2}\tau_{xy}\left(x,+\frac{b}{2},z\right)zdz$$

$$M_{yx}^{-}(x)=\int_{-H/2}^{H/2}\tau_{xy}\left(x,-\frac{b}{2},z\right)zdz$$

FUERZAS CORTANTES

Fuerzas cortantes en las caras (fuera del plano):

$$Q_x^+(y) = \int_{-H/2}^{H/2} \tau_{xz}\left(+\frac{a}{2}, y, z\right) dz$$

$$Q_x^-(y) = \int_{-H/2}^{H/2} \tau_{xz}\left(-\frac{a}{2}, y, z\right) dz$$

$$Q_y^+(x) = \int_{-H/2}^{H/2} \tau_{yz}\left(x, +\frac{b}{2}, z\right) dz$$

$$Q_y^-(x) = \int_{-H/2}^{H/2} \tau_{yz}\left(x, -\frac{b}{2}, z\right) dz$$

EQUILIBIO DE UN ELEMENTO DE PLACA

Ecuaciones de la estática:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

$$\sum M_x = 0$$

$$\sum M_y = 0$$

$$\sum M_z = 0$$

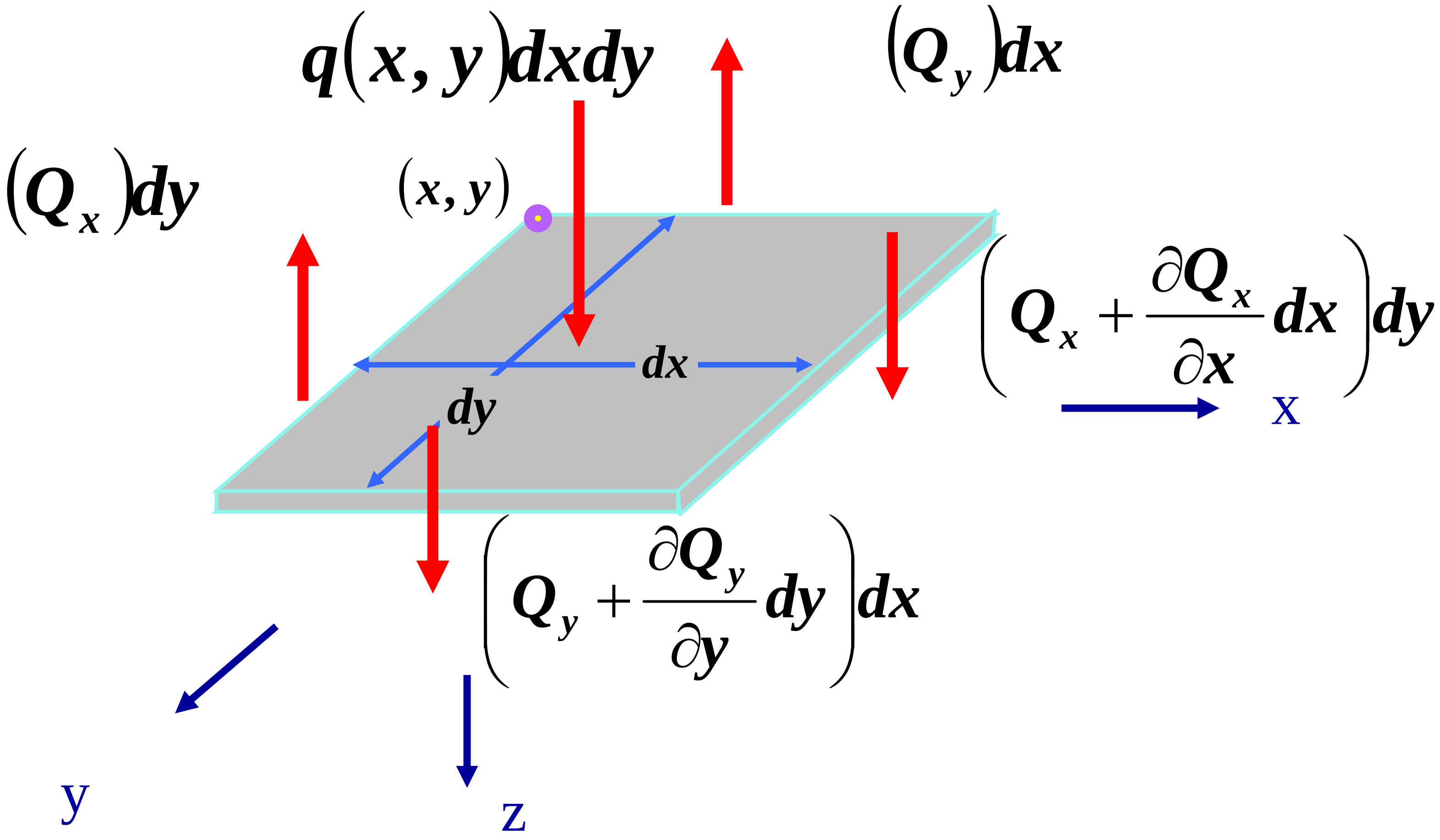
FUERA DEL PLANO DE LA PLACA

$$\sum F_z = 0$$

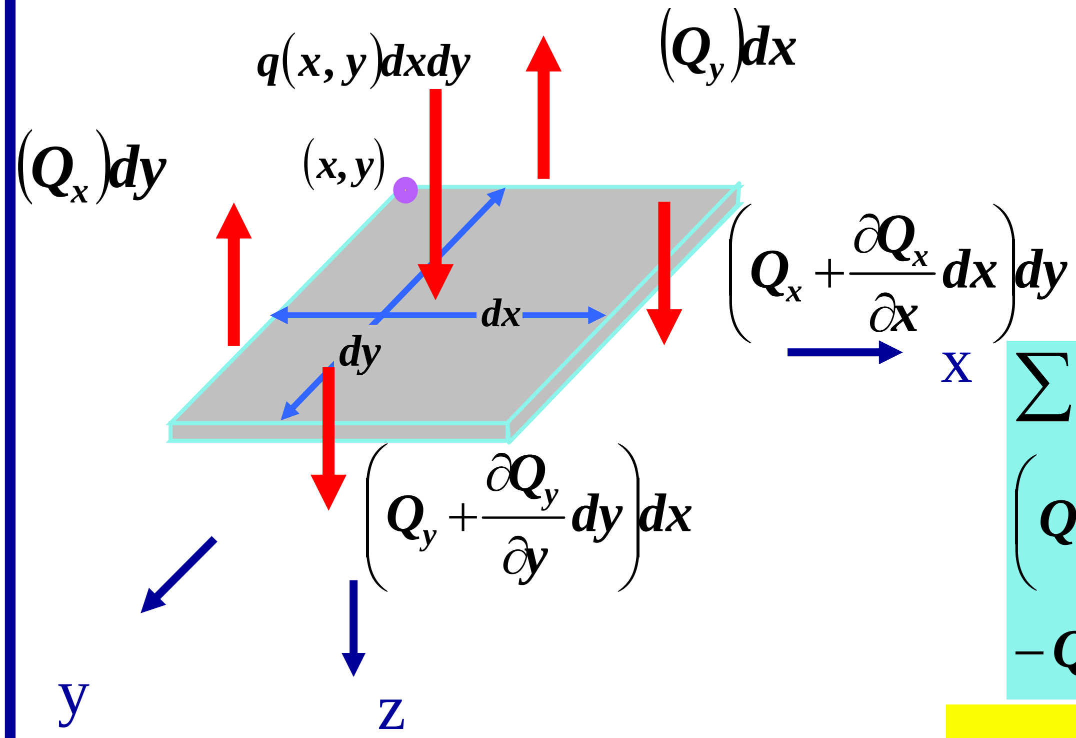
$$\sum M_x = 0$$

$$\sum M_y = 0$$

Fuerzas según el eje z:



EQUILIBRIO SEGÚN EL EJE z



$$\sum F_z = 0$$

$$\left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy + \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx -$$

$$-Q_x dy - Q_y dx + q \cdot dxdy$$

$$\frac{\partial \mathbf{Q}_x}{\partial \mathbf{x}} + \frac{\partial \mathbf{Q}_y}{\partial \mathbf{y}} + \mathbf{q} = 0$$

Ecuaciones de la estática:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

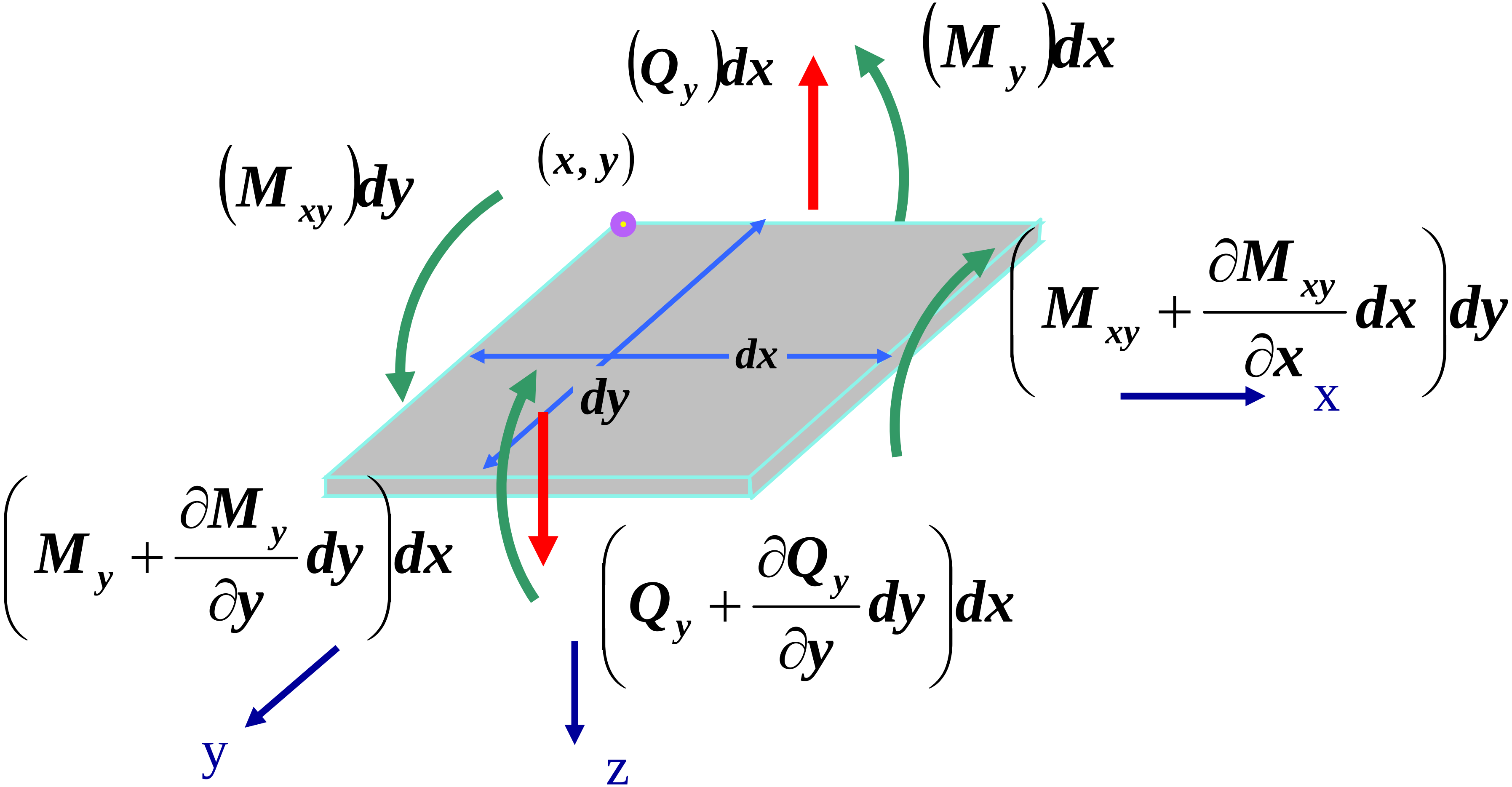
$$\frac{\partial \mathbf{Q}_x}{\partial \mathbf{x}} + \frac{\partial \mathbf{Q}_y}{\partial \mathbf{y}} + \mathbf{q} = \mathbf{0}$$

$$\sum M_x = 0$$

$$\sum M_y = 0$$

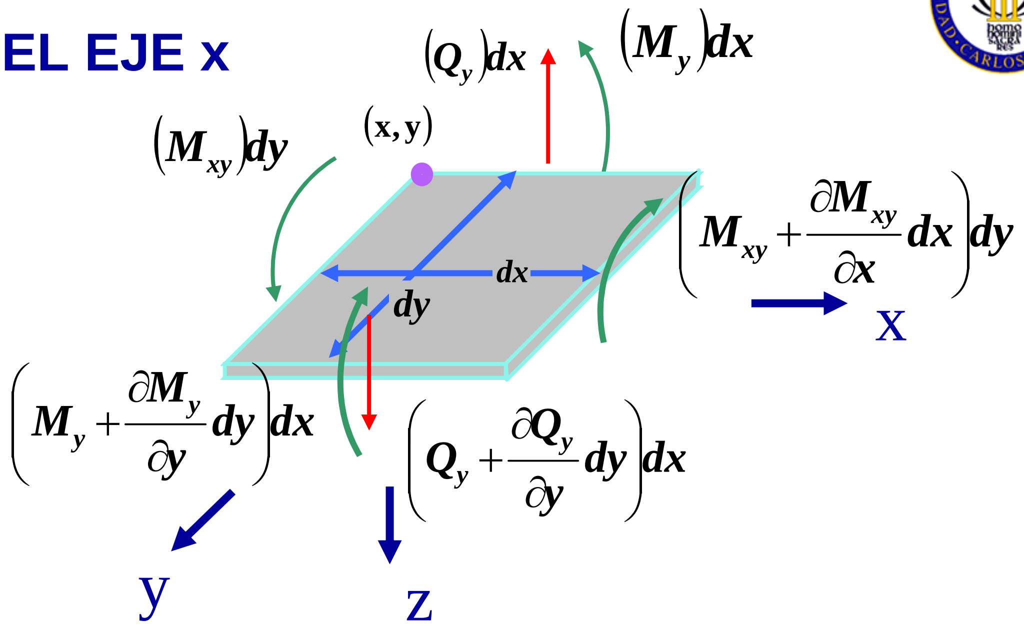
$$\sum M_z = 0$$

Momentos según el eje x



EQUILIBRIO ALREDEDOR DEL EJE x

(pasando por el centro del elemento)



$$\sum M_x = 0$$

$$(M_{xy})dy + (M_y)dy - \left(M_{xy}+\frac{\partial M_{xy}}{\partial x}dx\right)dy - \left(M_y+\frac{\partial M_y}{\partial y}dy\right)dx +$$

$$+\left(Q_y+\frac{\partial Q_y}{\partial y}dy\right)dx\cdot\frac{dy}{2} + (Q_y)dx\cdot\frac{dy}{2} = 0$$

EQUILIBRIO DE MOMENTOS ALREDEDOR DEL EJE x

$$\sum M_x = 0$$

$$\left(M_{xy}\right)dy + \left(M_y\right)dy - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}dx\right)dy - \left(M_y + \frac{\partial M_y}{\partial y}dy\right)dx +$$

$$+\left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx \cdot \frac{dy}{2} + \left(Q_y\right)dx \cdot \frac{dy}{2} = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = Q_y$$

Ecuaciones de la estática:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

$$\sum M_x = 0$$

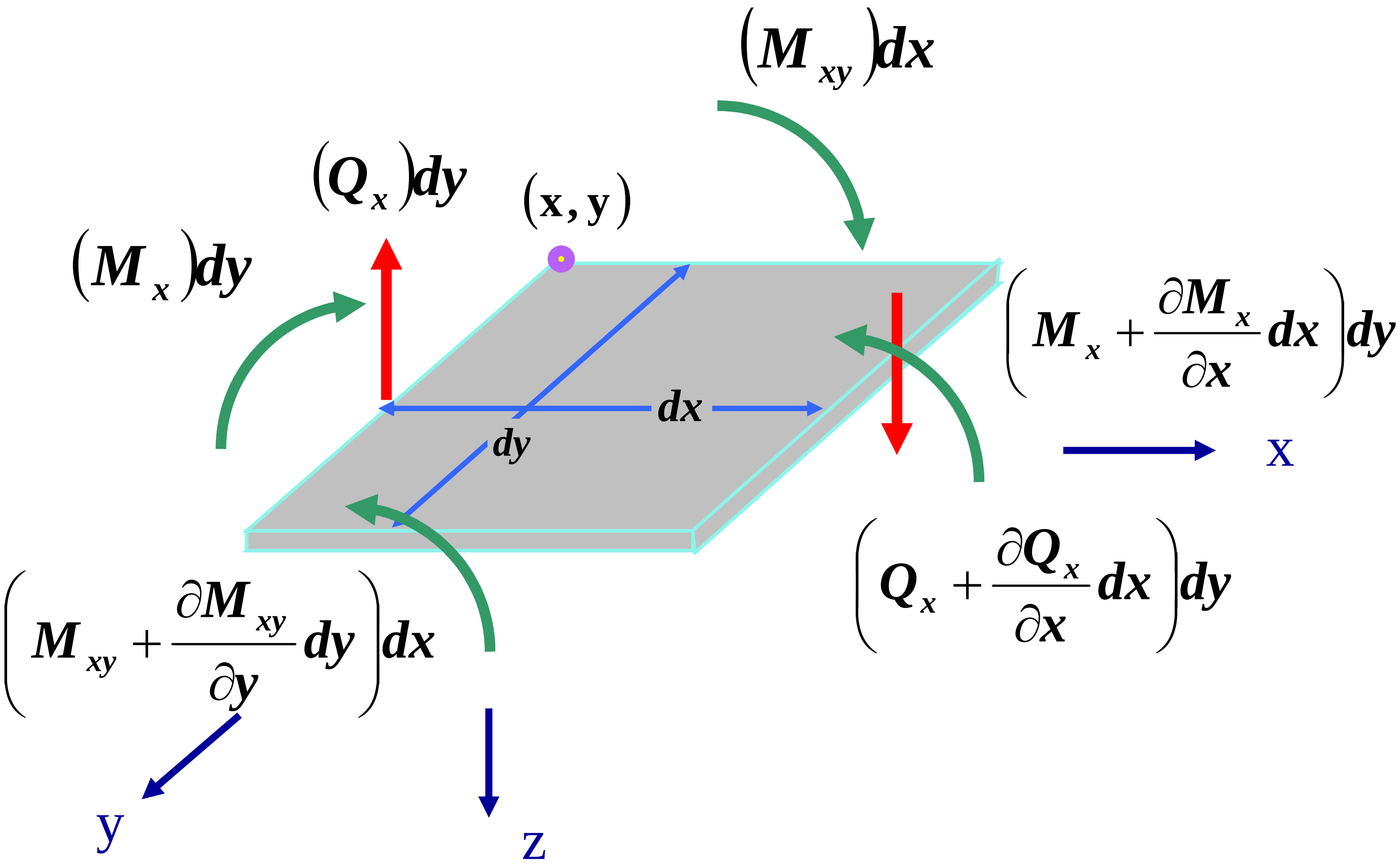
$$\sum M_y = 0$$

$$\sum M_z = 0$$

$$\frac{\partial \mathbf{Q}_x}{\partial \mathbf{x}} + \frac{\partial \mathbf{Q}_y}{\partial \mathbf{y}} + \mathbf{q} = \mathbf{0}$$

$$\frac{\partial \mathbf{M}_y}{\partial \mathbf{y}} + \frac{\partial \mathbf{M}_{xy}}{\partial \mathbf{x}} = \mathbf{Q}_y$$

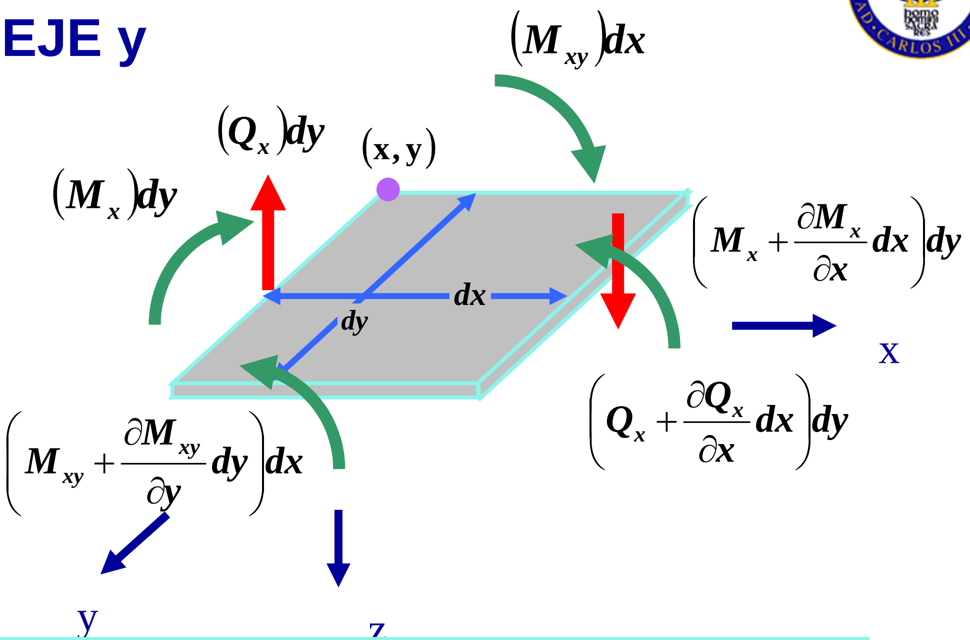
Momentos según el eje y:



EQUILIBRIO ALREDEDOR DEL EJE y

(pasando por el centro

del elemento)



$$\sum M_y = 0$$

$$-(M_{xy})dx - (M_x)dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial y}dy\right)dx + \left(M_x + \frac{\partial M_x}{\partial x}dx\right)dy +$$

$$-\left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy \cdot \frac{dx}{2} - (Q_x)dy \cdot \frac{dx}{2} = 0$$

EQUILIBRIO DE MOMENTOS ALREDEDOR DEL EJE y

$$\sum M_y = 0$$

$$-\left(M_{xy}\right)dx - \left(M_x\right)dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial y}dy\right)dx + \left(M_x + \frac{\partial M_x}{\partial x}dx\right)dy +$$

$$-\left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy \cdot \frac{dx}{2} - \left(Q_x\right)dy \cdot \frac{dx}{2} = 0$$

$$\frac{\partial \mathbf{M}_{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{M}_{\mathbf{xy}}}{\partial \mathbf{y}} = \mathbf{Q}_{\mathbf{x}}$$

Ecuaciones de la estática:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0 \qquad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

$$\sum M_x = 0 \qquad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = Q_y$$

$$\sum M_y = 0 \qquad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

$$\sum M_z = 0$$

RESUMEN DE ECUACIONES DE EQUILIBRIO (Fuera del plano):

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \qquad (1)$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \qquad (2)$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \qquad (3)$$

Si, en la primera de las ecuaciones anteriores, sustituimos Q_x y Q_y por sus expresiones (Ecs. 2 y 3):

$$\frac{\partial\left(\dfrac{\partial M_x}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}\right)}{\partial x}+\frac{\partial\left(\dfrac{\partial M_y}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}\right)}{\partial y}+q=0$$

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0$$

Ecuaciones de la estática:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

$$\sum M_x = 0$$

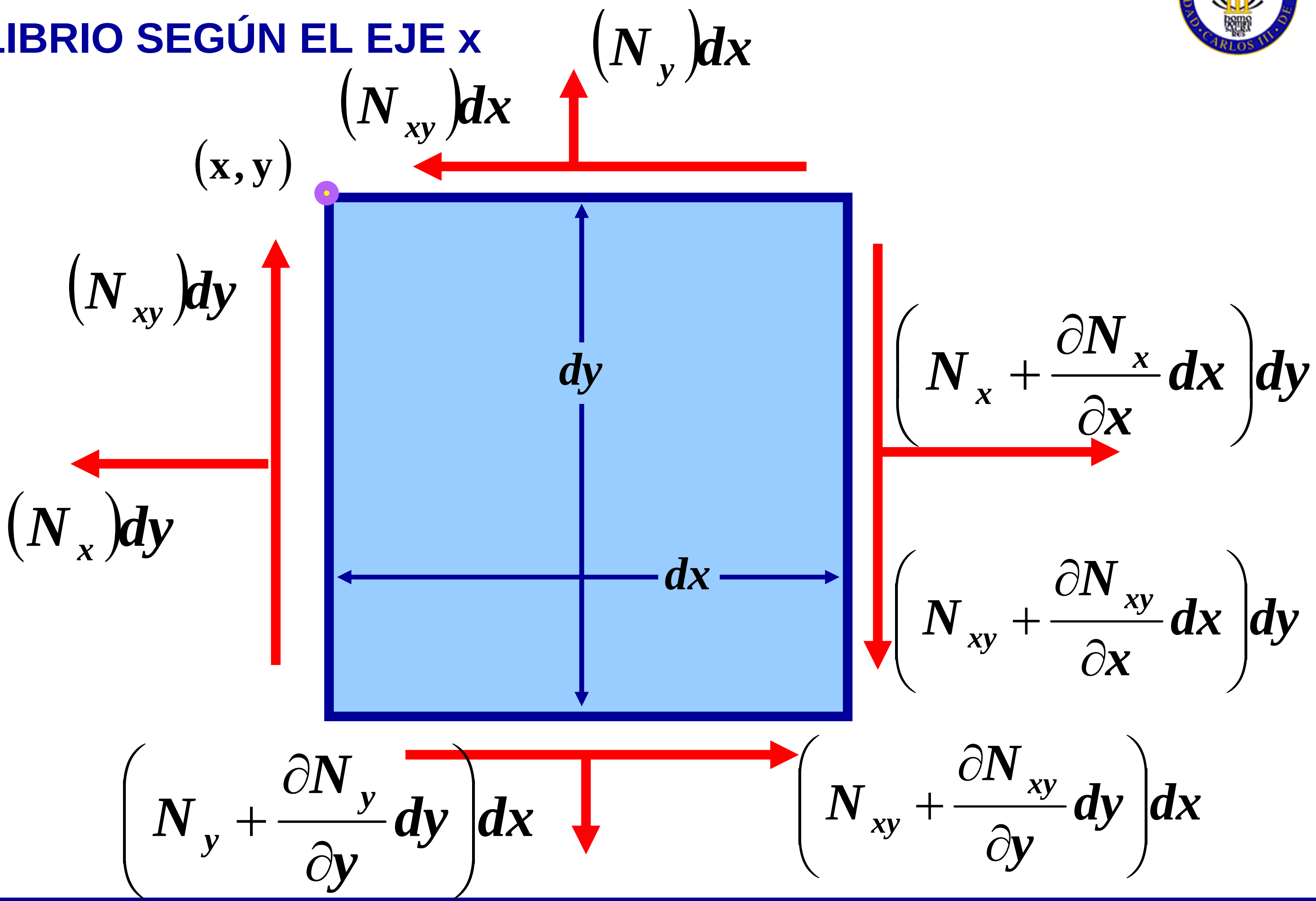
$$\sum M_y = 0$$

$$\sum M_z = 0$$

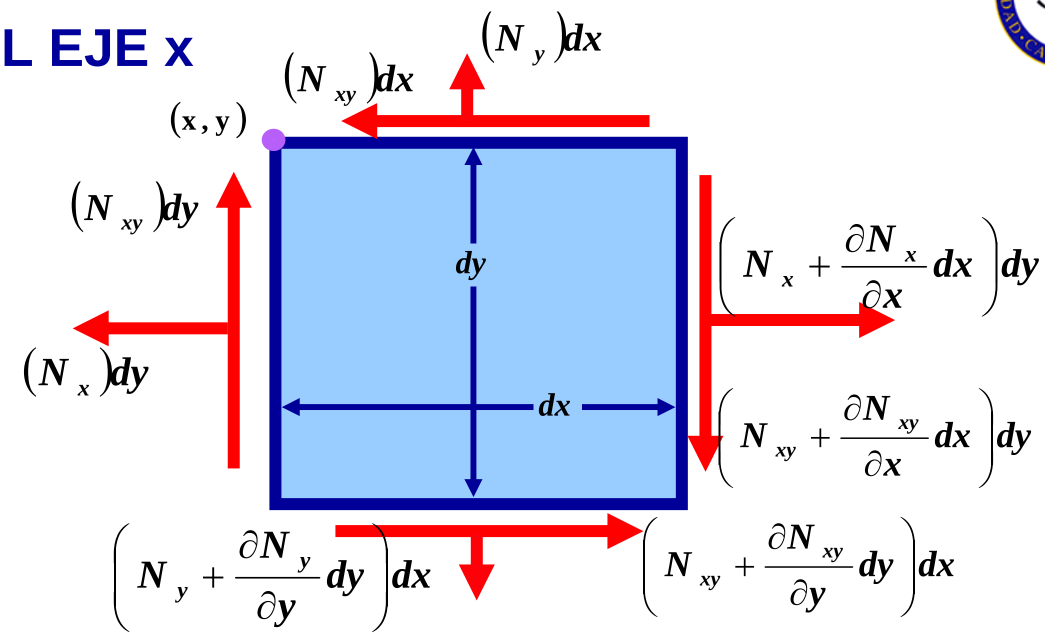
$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

EN EL PLANO DE LA PLACA

EQUILIBRIO SEGÚN EL EJE x



EQUILIBRIO SEGÚN EL EJE x



$$\sum F_x = 0$$

$$\left(N_x + \frac{\partial N_x}{\partial x}dx\right)dy + \left(N_{xy} + \frac{\partial N_{xy}}{\partial y}dy\right)dx -$$

$$- N_x dy - N_{xy} dx = 0$$

$$\sum F_x = 0$$

$$\left(N_x + \frac{\partial N_x}{\partial x} dx \right) dy + \left(N_{xy} + \frac{\partial N_{xy}}{\partial y} dy \right) dx -$$

$$- N_x dy - N_{xy} dx = 0$$

$$\frac{\partial \mathbf{N_x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N_{xy}}}{\partial \mathbf{y}} = \mathbf{0}$$

Ecuaciones de la estática:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum F_z = 0$$

$$\sum M_x = 0$$

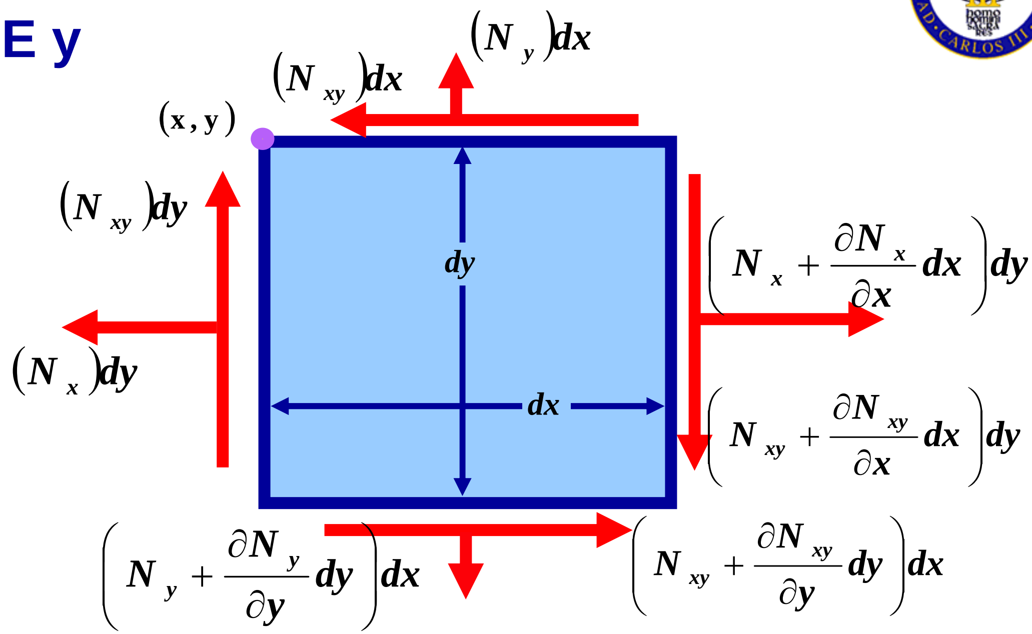
$$\sum M_y = 0$$

$$\sum M_z = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

EQUILIBRIO SEGÚN EL EJE y



$$\sum F_y = 0$$

$$\left(N_y + \frac{\partial N_y}{\partial y}dy\right)dx + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x}dx\right)dy -$$

$$- N_y dx - N_{xy} dy = 0$$

$$\sum F_y = 0$$

$$\left(N_y + \frac{\partial N_y}{\partial y} dy \right) dx + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} dx \right) dy -$$

$$- N_y dx - N_{xy} dy = 0$$

$$\frac{\partial \mathrm{N}_{\mathrm{xy}}}{\partial \mathrm{x}} + \frac{\partial \mathrm{N}_{\mathrm{y}}}{\partial \mathrm{y}} = \mathbf{0}$$

Ecuaciones de la estática:

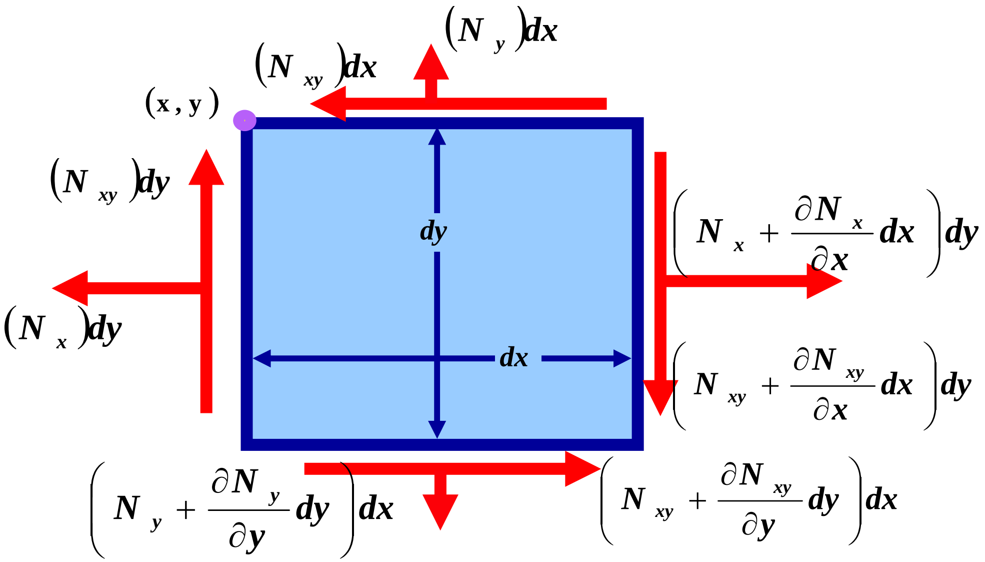
$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0$$
$$\sum M_x = 0$$
$$\sum M_y = 0$$
$$\sum M_z = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

EQUILIBRIO DE MOMENTOS ALREDEDOR DEL EJE z



$$\sum M_z = 0$$

$$-(N_{xy})dx \cdot \frac{dy}{2} + (N_{xy})dy \cdot \frac{dx}{2} - \left(N_{xy} + \frac{\partial N_{xy}}{\partial y} dy\right) dx \cdot \frac{dy}{2} +$$

$$+\left(N_{xy} + \frac{\partial N_{xy}}{\partial x} dx\right) dy \cdot \frac{dx}{2} = 0$$

$$\sum M_z = 0$$

$$-\left(N_{xy}\right)dx \cdot \frac{dy}{2} + \left(N_{xy}\right)dy \cdot \frac{dx}{2} - \left(N_{xy} + \frac{\partial N_{xy}}{\partial y}dy\right)dx \cdot \frac{dy}{2} +$$

$$+\left(N_{xy} + \frac{\partial N_{xy}}{\partial x}dx\right)dy \cdot \frac{dx}{2} = 0$$

¡Esta ecuación se satisface automáticamente!

Ecuaciones de la estática:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0$$
$$\sum M_x = 0$$
$$\sum M_y = 0$$
$$\sum M_z = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

¡Esta ecuación se satisface automáticamente!

ECUACIONES DE EQUILIBRIO DE LA PLACA

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

Y, entonces, ¿cómo se deducen los esfuerzos cortantes?

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \qquad (1)$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \qquad (2)$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \qquad (3)$$

CONDICIONES DE CONTORNO:

- $x = +a/2$
- $x = -a/2$
- $y = +b/2$
- $y = -b/2$

x = +a/2

Debe fijarse:

i. $N_x = N_x^+ \quad \text{ó} \quad u_0$

ii. $N_{xy} = N_{xy}^+ \quad \text{ó} \quad v_0$

iii. $\dfrac{\partial M_x}{\partial x} + 2\dfrac{\partial M_{xy}}{\partial y} = Q_x^+ + \dfrac{\partial M_{xy}^+}{\partial y} \quad \text{ó} \quad w_0$

iv. $M_x = M_x^+ \quad \text{ó} \quad \dfrac{\partial w_0}{\partial x}$

x = -a/2

Debe fijarse:

i. $N_x = N_x^- \quad \text{ó} \quad u_0$

ii. $N_{xy} = N_{xy}^- \quad \text{ó} \quad v_0$

iii. $\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} = Q_x^- + \frac{\partial M_{xy}^-}{\partial y} \quad \text{ó} \quad w_0$

iv. $M_x = M_x^- \quad \text{ó} \quad \frac{\partial w_0}{\partial x}$

y = +b/2

Debe fijarse:

i. $N_y = N_y^+ \quad$ ó $\quad v_0$

ii. $N_{xy} = N_{yx}^+ \quad$ ó $\quad u_0$

iii. $\dfrac{\partial M_y}{\partial y} + 2\dfrac{\partial M_{xy}}{\partial x} = Q_y^+ + \dfrac{\partial M_{yx}^+}{\partial y} \quad$ ó $\quad w_0$

iv. $M_y = M_y^+ \quad$ ó $\quad \dfrac{\partial w_0}{\partial y}$

y = -b/2

Debe fijarse:

i. $N_y = N_y^- \quad ó \quad v_0$

ii. $N_{xy} = N_{yx}^- \quad ó \quad u_0$

iii. $\dfrac{\partial M_y}{\partial y} + 2\dfrac{\partial M_{xy}}{\partial x} = Q_y^- + \dfrac{\partial M_{yx}^-}{\partial y} \quad ó \quad w_0$

iv. $M_y = M_y^- \quad ó \quad \dfrac{\partial w_0}{\partial y}$

DEFORMACIONES EN EL PLANO MEDIO:

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$

CURVATURAS:

$$\kappa_x = -\frac{\partial^2 w_0}{\partial x^2}$$

$$\kappa_y = -\frac{\partial^2 w_0}{\partial y^2}$$

$$\kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y}$$

ECUACIONES BASADAS EN DESPLAZAMIENTOS:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$

$$\kappa_x = -\frac{\partial^2 w_0}{\partial x^2}$$

$$\kappa_y = -\frac{\partial^2 w_0}{\partial y^2}$$

$$\kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y}$$

$$N_x = A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + A_{16}\gamma_{xy}^0 + B_{11}\kappa_x + B_{12}\kappa_y + B_{16}\kappa_{xy}$$

$$N_x = A_{11}\frac{\partial u_0}{\partial x} + A_{12}\frac{\partial v_0}{\partial y} + A_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - B_{11}\frac{\partial^2 w_0}{\partial x^2} - B_{12}\frac{\partial^2 w_0}{\partial y^2} - 2B_{16}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

$$N_y = A_{12}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + A_{26}\gamma_{xy}^0 + B_{12}\kappa_x + B_{22}\kappa_y + B_{26}\kappa_{xy}$$

$$N_y = A_{12}\frac{\partial u_0}{\partial x} + A_{22}\frac{\partial v_0}{\partial y} + A_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - B_{12}\frac{\partial^2 w_0}{\partial x^2} - B_{22}\frac{\partial^2 w_0}{\partial y^2} - 2B_{26}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

ECUACIONES BASADAS EN DESPLAZAMIENTOS:

$$N_{xy} = A_{16}\varepsilon_x^0 + A_{26}\varepsilon_y^0 + A_{66}\gamma_{xy}^0 + B_{16}\kappa_x + B_{26}\kappa_y + B_{66}\kappa_{xy}$$

$$N_{xy} = A_{16}\frac{\partial u_0}{\partial x} + A_{26}\frac{\partial v_0}{\partial y} + A_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - B_{16}\frac{\partial^2 w_0}{\partial x^2} - B_{26}\frac{\partial^2 w_0}{\partial y^2} - 2B_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

ECUACIONES BASADAS EN DESPLAZAMIENTOS:

$$M_x = B_{11}\varepsilon_x^0 + B_{12}\varepsilon_y^0 + B_{16}\gamma_{xy}^0 + D_{11}\kappa_x + D_{12}\kappa_y + D_{16}\kappa_{xy}$$

$$M_x = B_{11}\frac{\partial u_0}{\partial x} + B_{12}\frac{\partial v_0}{\partial y} + B_{16}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{11}\frac{\partial^2 w_0}{\partial x^2} - D_{12}\frac{\partial^2 w_0}{\partial y^2} - 2D_{16}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

ECUACIONES BASADAS EN DESPLAZAMIENTOS:

$$M_y = B_{12}\varepsilon_x^0 + B_{22}\varepsilon_y^0 + B_{26}\gamma_{xy}^0 + D_{12}\kappa_x + D_{22}\kappa_y + D_{26}\kappa_{xy}$$

$$M_y = B_{12}\frac{\partial u_0}{\partial x} + B_{22}\frac{\partial v_0}{\partial y} + B_{26}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{12}\frac{\partial^2 w_0}{\partial x^2} - D_{22}\frac{\partial^2 w_0}{\partial y^2} - 2D_{26}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

ECUACIONES BASADAS EN DESPLAZAMIENTOS:

$$M_{xy} = B_{16}\varepsilon_x^0 + B_{26}\varepsilon_y^0 + B_{66}\gamma_{xy}^0 + D_{16}\kappa_x + D_{26}\kappa_y + D_{66}\kappa_{xy}$$

$$M_{xy} = B_{16}\frac{\partial u_0}{\partial x} + B_{26}\frac{\partial v_0}{\partial y} + B_{66}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - D_{16}\frac{\partial^2 w_0}{\partial x^2} - D_{26}\frac{\partial^2 w_0}{\partial y^2} - 2D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)$$

ECUACIONES DE EQUILIBRIO DE LA PLACA:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

ECUACIONES BASADAS EN CONDICIONES DE EQUILIBRIO:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$N_x = A_{11}\frac{\partial u^0}{\partial x} + A_{12}\frac{\partial v^0}{\partial y} + A_{16}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - B_{11}\frac{\partial^2 w^0}{\partial x^2} - B_{12}\frac{\partial^2 w^0}{\partial y^2} - 2B_{16}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

$$N_{xy} = A_{16}\frac{\partial u^0}{\partial x} + A_{26}\frac{\partial v^0}{\partial y} + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) - B_{16}\frac{\partial^2 w^0}{\partial x^2} - B_{26}\frac{\partial^2 w^0}{\partial y^2} - 2B_{66}\left(\frac{\partial^2 w^0}{\partial x \partial y}\right)$$

Sustituyendo las expresiones de N_x y N_{xy} en:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2} + A_{12}\frac{\partial^2 v^0}{\partial y \partial x} + A_{16}\left(\frac{\partial^2 u^0}{\partial y \partial x} + \frac{\partial^2 v^0}{\partial x^2}\right)$$

$$- B_{11}\frac{\partial^3 w^0}{\partial x^3} - B_{12}\frac{\partial^3 w^0}{\partial x \partial y^2} - 2B_{16}\left(\frac{\partial^3 w^0}{\partial x^2 \partial y}\right)$$

$$+ A_{16}\frac{\partial^2 u^0}{\partial x \partial y} + A_{26}\frac{\partial^2 v^0}{\partial y^2} + A_{66}\left(\frac{\partial^2 u^0}{\partial y^2} + \frac{\partial^2 v^0}{\partial x \partial y}\right)$$

$$- B_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y} - B_{26}\frac{\partial^3 w^0}{\partial y^3} - 2B_{66}\left(\frac{\partial^3 w^0}{\partial x \partial y^2}\right) = 0$$

Reorganizando la última expresión:

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}$$

$$+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

En definitiva:

$$\frac{\partial \mathbf{N}_{\mathbf{x}}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N}_{\mathbf{xy}}}{\partial \mathbf{y}} = \mathbf{0}$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2} + 2A_{16}\frac{\partial^2 u^0}{\partial x \partial y} + A_{66}\frac{\partial^2 u^0}{\partial y^2}$$

$$+ A_{16}\frac{\partial^2 v^0}{\partial x^2} + \left(A_{12} + A_{66}\right)\frac{\partial^2 v^0}{\partial x \partial y} + A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$- B_{11}\frac{\partial^3 w^0}{\partial x^3} - 3B_{16}\frac{\partial^3 w^0}{\partial x^2 \partial y} - \left(B_{12} + 2B_{66}\right)\frac{\partial^3 w^0}{\partial x \partial y^2} - B_{26}\frac{\partial^3 w^0}{\partial y^3} = 0$$

Ecuación diferencial 1

De la misma forma:

$$\frac{\partial \mathbf{N}_{xy}}{\partial \mathbf{x}} + \frac{\partial \mathbf{N}_{y}}{\partial \mathbf{y}} = \mathbf{0}$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2} + \left(A_{12} + A_{66}\right)\frac{\partial^2 u^0}{\partial x \partial y} + A_{26}\frac{\partial^2 u^0}{\partial y^2}$$

$$+ A_{66}\frac{\partial^2 v^0}{\partial x^2} + 2A_{26}\frac{\partial^2 v^0}{\partial x \partial y} + A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$- B_{16}\frac{\partial^3 w^0}{\partial x^3} - \left(B_{12} + 2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 w^0}{\partial x \partial y^2} - B_{22}\frac{\partial^3 w^0}{\partial y^3} = 0$$

Ecuación diferencial 2

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

$$\begin{aligned}
&D_{11}\frac{\partial^4 w^0}{\partial x^4} + 4D_{16}\frac{\partial^4 w^0}{\partial x^3 \partial y} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2 \partial y^2} \\
&+ 4D_{26}\frac{\partial^4 w^0}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w^0}{\partial y^4} \\
&- B_{11}\frac{\partial^3 u^0}{\partial x^3} - 3B_{16}\frac{\partial^3 u^0}{\partial x^2 \partial y} - \left(B_{12} + 2B_{66}\right)\frac{\partial^3 u^0}{\partial x \partial y^2} - B_{26}\frac{\partial^3 u^0}{\partial y^3} \\
&- B_{16}\frac{\partial^3 v^0}{\partial x^3} - \left(B_{12} + 2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2 \partial y} - 3B_{26}\frac{\partial^3 v^0}{\partial x \partial y^2} - B_{22}\frac{\partial^3 v^0}{\partial y^3} = q
\end{aligned}$$

En definitiva, hemos llegado a plantear tres ecuaciones diferenciales (Ecuaciones 1, 2 y 3) en las que aparecen, como incógnitas, u^0, v^0 y w^0 que, una vez resueltas y verificando las condiciones de contorno, nos permitirían calcular el campo de desplazamientos dentro del laminado. de este último, podríamos determinar las deformaciones en cada punto del laminado y, de este último, el campo tensional.

LAMINADOS SIMÉTRICOS:

$$\mathbf{B_{ij} = 0}$$

Ecuaciones generales:

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial y\partial x}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial y\partial x}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$
$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial y\partial x}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial y\partial x}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$
$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$
$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$
$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

Ecuaciones para el caso de laminados simétricos:

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}=0$$

LAMINADOS SIMÉTRICOS BALANCEADOS:

$$B_{ij} = 0$$

$$A_{16} = 0$$

$$A_{26} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$

$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

Ecuaciones para el caso de laminados simétricos balanceados:

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x \partial y}=0$$

$$\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x \partial y}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+A_{22}\frac{\partial^2 v^0}{\partial y^2}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3 \partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2 \partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x \partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}=q$$

LAMINADOS SIMÉTRICOS DE LÁMINAS CRUZADAS:

$$B_{ij} = 0$$

$$A_{16} = 0 \qquad D_{16} = 0$$

$$A_{26} = 0 \qquad D_{26} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$

$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

Ecuaciones para el caso de laminados simétricos de láminas cruzadas:

$$\mathbf{A_{11}\frac{\partial^2 u^0}{\partial x^2} + A_{66}\frac{\partial^2 u^0}{\partial y^2} + \left(A_{12} + A_{66}\right)\frac{\partial^2 v^0}{\partial x \partial y} = 0}$$

$$\mathbf{\left(A_{12} + A_{66}\right)\frac{\partial^2 u^0}{\partial x \partial y} + A_{66}\frac{\partial^2 v^0}{\partial x^2} + A_{22}\frac{\partial^2 v^0}{\partial y^2} = 0}$$

$$\mathbf{D_{11}\frac{\partial^4 w^0}{\partial x^4 \partial y} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w^0}{\partial y^4} = q}$$

PLACAS ISÓTROPAS:

$$\boldsymbol{B}_{ij} = 0$$

$$\boldsymbol{A}_{11} = \boldsymbol{A}_{22} = \frac{\boldsymbol{EH}}{1-\nu^2} = \boldsymbol{A} \qquad \boldsymbol{A}_{12} = \boldsymbol{A}_{21} = \nu\frac{\boldsymbol{EH}}{1-\nu^2} = \nu\boldsymbol{A}$$

$$\boldsymbol{A}_{66} = \frac{\boldsymbol{EH}}{2(1+\nu)} = \frac{1-\nu}{2}\boldsymbol{A} \qquad \boldsymbol{A}_{16} = \boldsymbol{A}_{26} = 0$$

$$\boldsymbol{D}_{11} = \boldsymbol{D}_{22} = \frac{\boldsymbol{EH}^3}{12(1-\nu^2)} = \boldsymbol{D} \qquad \boldsymbol{D}_{12} = \boldsymbol{D}_{21} = \frac{\nu\boldsymbol{EH}^3}{12(1-\nu^2)} = \nu\boldsymbol{D}$$

$$\boldsymbol{D}_{66} = \frac{\boldsymbol{EH}^3}{24(1+\nu)} = \frac{1-\nu}{2}\boldsymbol{D} \qquad \boldsymbol{D}_{16} = \boldsymbol{D}_{26} = 0$$

$$A_{11}\frac{\partial^2 u^0}{\partial x^2}+2A_{16}\frac{\partial^2 u^0}{\partial x\partial y}+A_{66}\frac{\partial^2 u^0}{\partial y^2}+A_{16}\frac{\partial^2 v^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 v^0}{\partial x\partial y}+A_{26}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{11}\frac{\partial^3 w^0}{\partial x^3}-3B_{16}\frac{\partial^3 w^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$A_{16}\frac{\partial^2 u^0}{\partial x^2}+\left(A_{12}+A_{66}\right)\frac{\partial^2 u^0}{\partial x\partial y}+A_{26}\frac{\partial^2 u^0}{\partial y^2}+A_{66}\frac{\partial^2 v^0}{\partial x^2}+2A_{26}\frac{\partial^2 v^0}{\partial x\partial y}+A_{22}\frac{\partial^2 v^0}{\partial y^2}$$

$$-B_{16}\frac{\partial^3 w^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w^0}{\partial y^3}=0$$

$$D_{11}\frac{\partial^4 w^0}{\partial x^4}+4D_{16}\frac{\partial^4 w^0}{\partial x^3\partial y}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w^0}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w^0}{\partial x\partial y^3}+D_{22}\frac{\partial^4 w^0}{\partial y^4}$$

$$-B_{11}\frac{\partial^3 u^0}{\partial x^3}-3B_{16}\frac{\partial^3 u^0}{\partial x^2\partial y}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 u^0}{\partial x\partial y^2}-B_{26}\frac{\partial^3 u^0}{\partial y^3}$$

$$-B_{16}\frac{\partial^3 v^0}{\partial x^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 v^0}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v^0}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v^0}{\partial y^3}=q$$

Ecuaciones para placas isótropas:

$$A\frac{\partial^2 u^0}{\partial x^2}+\left(\frac{1-\nu}{2}\right)A\frac{\partial^2 u^0}{\partial y^2}+\left(\nu A+\frac{1-\nu}{2}A\right)\frac{\partial^2 v^0}{\partial y \partial x}=0$$

$$\frac{\partial^2 u^0}{\partial x^2}+\left(\frac{1-\nu}{2}\right)\frac{\partial^2 u^0}{\partial y^2}+\left(\nu+\frac{1-\nu}{2}\right)\frac{\partial^2 v^0}{\partial y \partial x}=0$$

$$\frac{\partial^2 u^0}{\partial x^2}+\left(\frac{1-\nu}{2}\right)\frac{\partial^2 u^0}{\partial y^2}+\left(\frac{1+\nu}{2}\right)\frac{\partial^2 v^0}{\partial y \partial x}=0$$

$$\frac{\partial^2 \boldsymbol{u}^0}{\partial \boldsymbol{x}^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 \boldsymbol{u}^0}{\partial \boldsymbol{y}^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 \boldsymbol{v}^0}{\partial \boldsymbol{y}\partial \boldsymbol{x}} = 0$$

$$\left(\frac{1+\nu}{2}\right)\frac{\partial^2 \boldsymbol{u}^0}{\partial \boldsymbol{y}\partial \boldsymbol{x}} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 \boldsymbol{v}^0}{\partial \boldsymbol{x}^2} + \frac{\partial^2 \boldsymbol{v}^0}{\partial \boldsymbol{y}^2} = 0$$

$$\frac{\partial^4 \boldsymbol{w}^0}{\partial \boldsymbol{x}^4} + 2\frac{\partial^4 \boldsymbol{w}^0}{\partial \boldsymbol{x}^2 \partial \boldsymbol{y}^2} + \frac{\partial^4 \boldsymbol{w}^0}{\partial \boldsymbol{y}^4} = \frac{\boldsymbol{q}}{\boldsymbol{D}}$$

CONSTANTE DE RIGIDEZ A FLEXIÓN DE UNA PLACA

$$D = \frac{EH^3}{12 \cdot (1 - \nu^2)}$$