Universidad Carlos III de Madrid

Escuela Politécnica Superior

Departamento de Matemáticas

First Course. Telecommunication Engineering

Calculus II. First Test, March 2nd, 2010

GRADE

Surname .... Name ....

D.N.I. .... Group ....

Time length: 80 min.

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1. Find the following antiderivatives

$$\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx, \qquad \int \sin^4 x \cos^2 x\,dx, \qquad \int (2x+1)\log x\,dx.$$

[4.5 p.]

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2. Compute the limit

$$\lim_{x\to 0} \frac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x te^{t^2}\,dt}.$$

[2 p.]

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3. Analyze the convergence of the following improper integral for the different values of a

$$\int_0^\infty \frac{1}{x^2+a^2}\,dx.$$

[2 p.]

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4. Compute the volume formed by revolving about the x axis the region between the function $y = e^{-x} + 2$ and its asymptote for $x \in [0,\infty)$.

[1.5 p.]

ANSWERS:

1. a) Change of variables: $t^2 = 1 + \sqrt[3]{x} \to x = (t^2-1)^3 \to dx = 6t(t^2-1)^2\,dt$

$$\begin{aligned}\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx &= \int \frac{t}{t^2-1} 6t(t^2-1)^2\,dt = \int 6t^2(t^2-1)\,dt = \frac{6}{5}t^5 - 2t^3 + C \\ &= \frac{6}{5}(1+\sqrt[3]{x})^{\frac{5}{2}} - 2(1+\sqrt[3]{x})^{\frac{3}{2}} + C.\end{aligned}$$

(b) Using double angle formulas:

$$\begin{aligned}\int \sin^4 x \cos^2 x\,dx &= \int [\sin^2 x]^2 \cos^2 x\,dx = \int \left[\frac{1-\cos 2x}{2}\right]^2 \left[\frac{1+\cos 2x}{2}\right] dx = \\ &= \frac{1}{8}\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)dx \\ &= \frac{1}{8}\left[x - \frac{\sin 2x}{2} - \int \cos^2 2x dx + \int \cos^3 2x dx\right]\end{aligned}$$

Computing the last integrals

$$\begin{aligned}\int \cos^2 2x\,dx &= \int \frac{1+\cos 4x}{2}\,dx = \frac{x}{2} + \frac{\sin 4x}{8} \\ \int \cos^3 2x\,dx &= \int (1-\sin^2 2x)\cos 2x\,dx = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}\end{aligned}$$

Finally

$$\begin{aligned}\int \cos^4 x \sin^2 x\,dx &= \frac{1}{8}\left[x - \frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8} + \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}\right] + C = \\ &= \frac{1}{8}\left[\frac{x}{2} - \frac{\sin 4x}{8} - \frac{\sin^3 2x}{6}\right] + C.\end{aligned}$$

(c) Integrating by parts $\begin{cases} u = logx \to du = \dfrac{1}{x}\,dx \\ \\ dv = (2x+1)\,dx \to v = x^2 + x \end{cases}$

$$\int (2x+1)\log x\,dx = \left(x^2+x\right) logx - \int \frac{\left(x^2+x\right)}{x}\,dx = \left(x^2+x\right) logx - \frac{x^2}{2} - x + C.$$

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2. We have an $\left[\frac{0}{0}\right]$ indeterminate form, so we apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x te^{t^2}\,dt} \underset{LH}{=} \lim_{x\to 0} \frac{2\left(\int_0^x e^{t^2}\,dt\right)e^{x^2}}{xe^{x^2}} = \lim_{x\to 0} \frac{2\left(\int_0^x e^{t^2}\,dt\right)}{x} \underset{LH}{=} \lim_{x\to 0} \frac{2e^{x^2}}{1} = 2.$$

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3. Let us separate the integral in two parts:

$$\int_0^\infty \frac{1}{x^2+a^2}\,dx. = \int_0^1 \frac{1}{x^2+a^2}\,dx. + \int_1^\infty \frac{1}{x^2+a^2}\,dx.$$

- For $a = 0$ the first integral diverges:

$$a = 0 \Rightarrow \int_0^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0^+} \left[-\frac{1}{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0^+} \left[-1+\frac{1}{\varepsilon}\right] = \infty.$$

  Thus $\int_0^\infty \frac{1}{x^2+a^2}\,dx$ **is divergent for** $a = 0$**.**

- Para $a \neq 0$ the first integral is not improper, so we just have to study the second one. As the second integral is only improper at ∞, we can compare it with $\frac{1}{x^2}$:

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+a^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^2}{x^2+a^2} = 1.$$

  This implies that both $\int_1^\infty \frac{1}{x^2+a^2}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$ have the same behaviour.

  As $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2}\,dx = \lim_{N\to\infty} \left[-\frac{1}{x}\right]_1^N = 1 - \lim_{N\to\infty} \frac{1}{N} = 1$, we conclude that our integral **is convergent for** $a \neq 0$**.**

  We can also prove the last assertion just by computing directly the integral:

  $\lim_{N\to\infty} \int_1^N \frac{1}{x^2+a^2} = \lim_{N\to\infty} \frac{1}{a} \arctan\left(\frac{x}{a}\right)\Big|_0^N = \frac{\pi}{2a}$, so it is convergent.

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3. As $\lim_{x\to\infty} \left[e^{-x}+2\right] = 2$ the function has an horizontal asymptote at $y = 2$. So the volume is

$$V = \int_0^\infty \pi\left[y(x)^2 - 2^2\right]\,dx = \int_0^\infty \pi\left[(e^{-x}+2)^2 - 2^2\right]\,dx = \int_0^\infty \pi\left[e^{-2x} + 4e^{-x} + 4 - 4\right]\,dx =$$

$$= \int_0^\infty \pi\left[e^{-2x} + 4e^{-x}\right]\,dx = \pi\left[-\frac{e^{-2x}}{2} - 4e^{-x}\right]_0^\infty = \pi\left[\frac{1}{2}+4\right] = \frac{9\pi}{2}.$$