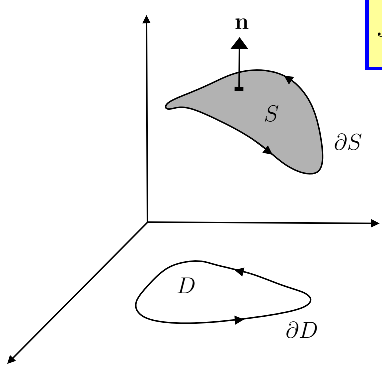
Units 4 & 5

Theorems of Vector Analysis

Theorem (Stokes' Theorem). *Let S be an oriented surface defined by a one-to-one parametrization preserving the orientation, $\boldsymbol{\phi}\colon D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$. Let ∂S denote the oriented boundary of S and $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field on S. Then,*

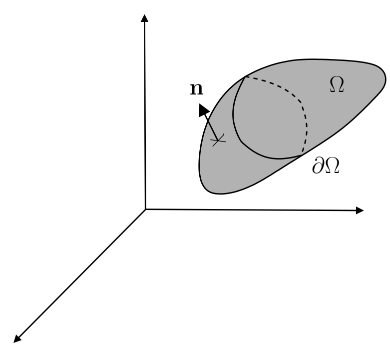
$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$



The orientation on ∂S is the orientation induced by the upward normal $\mathbf{n}$ of S. When you walk along the boundary ∂S, with the normal $\mathbf{n}$ as your upright direction, the surface S must be on your left.

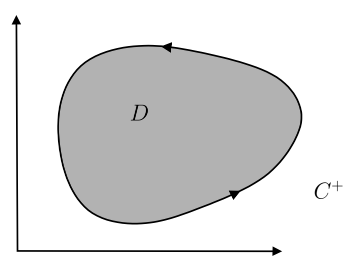
Theorem (Gauss' Divergence Theorem). *Let Ω be a solid region in $\mathbb{R}^3$. Denote by $\partial\Omega$ the oriented closed surface that bounds Ω. Let $\mathbf{F}\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field on Ω. Then,*

$$\iiint_\Omega div\,\mathbf{F}\, dv = \iint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}.$$



Theorem (Green's Theorem).
Let D be a simply connected region on $\mathbb{R}^2$ and let C be its boundary. Suppose $P, Q\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ are C^1. Then,

$$\int_{C^+} Pdx + Qdy \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$



where C^+ is the boundary of D with positive orientation.

Theorem. *If C is a simple closed curve that bounds a region to which Green's Theorem applies, then the area of the region D bounded by $C = \partial D$ is*

$$A = \frac{1}{2}\int_{\partial D} xdy - ydx.$$

Note. If the region is not simply connected, we break the region into simply connected regions and apply the theorem to each of them.