Bachelor in Statistics and Business
Universidad Carlos III de Madrid

Mathematical Methods II
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Name ______________________ Student number ______________________

SOLUTION: FINAL EXAM

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Instructions

- You have 3 hours to answer the final exam.
- Marks per question are given in bold. **Total marks: 10 points**
- The mark of this exam corresponds to the 60% of this course.
- Write your name in all the sheets.
- It is not allowed to use lecture notes, scientific calculators or cellphones during the exam.

1. Consider the following system of linear equations depending on the parameter h:

$$\begin{cases} 4x & & +z & = & 2 \\ -x & +y & +2z & = & 0 \\ & -4y & -9z & = & h \end{cases}$$

(a) Describe the solution set in terms of the parameter h. **[1]**

Solution: To describe the solution set of a system of linear equations we must obtain a matrix in echelon form equivalent to the augmented matrix of the system. The augmented matrix of the given system is:

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ -1 & 1 & 2 & 0 \\ 0 & -4 & -9 & h \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 4 & 0 & 1 & 2 \\ 0 & -4 & -9 & h \end{pmatrix} \underset{4R_1 + R_2 \to R_2}{\sim}$$

$$\begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 9 & 2 \\ 0 & -4 & -9 & h \end{pmatrix} \underset{R_2 + R_3 \to R_3}{\sim} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 9 & 2 \\ 0 & 0 & 0 & h+2 \end{pmatrix}$$

The system is inconsistent if the last column of the augmented matrix is a pivot column, that is, $h + 2 \neq 0$. Hence, for $h \neq -2$ the system is inconsistent.

The system is consistent if the last column of the augmented matrix of the system is not a pivot column, that is, $h+2=0$. Note that for $h=-2$, the system has a free variable which corresponds with z. Thus for $h=-2$ the system is consistent with infinitely many solutions.

There are no values of h such that the system is consistent with a unique solution.

(b) For that or those values of h that make the system consistent:

i. Write the solution set in parametric vector form. **[0.5]**

ii. Is that solution set a subspace of $\mathbb{R}^3$? **[0.5]**

Solution: For $h=-2$, we obtain the matrix in reduced echelon form equivalent to the augmented matrix of the system to describe the solution set:

$$\begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{-\frac{1}{4}R_2+R_1\to R_1}{\sim} \begin{pmatrix} -1 & 0 & -1/4 & -1/2 \\ 0 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{-R_1\to R_1 \\ R_2/4\to R_2}}{\sim} \begin{pmatrix} 1 & 0 & 1/4 & 1/2 \\ 0 & 1 & 9/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution set in parametric vector form is as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 - z/4 \\ 1/2 - 9z/4 \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1/4 \\ -9/4 \\ 1 \end{pmatrix}.$$

The solution set of the system for $h=-2$ is not a subspace because $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ does not belong to the solution set, that is, there are no values for z such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 - z/4 \\ 1/2 - 9z/4 \\ z \end{pmatrix}.$$

This system is inconsistent because we have $z=0$, and then $0=1/2$, what is impossible.

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2. Let T be a transformation between vector spaces defined by

$$\begin{aligned} T\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x,y) &\longmapsto (2x+3y, -y, -x+2y) \end{aligned}$$

(a) Check that the transformation T is linear. **[0.75]**

Solution: We must prove the following three properties:

$$T(0,0) = (0,0,0), \quad \text{OK.}$$

Moreover, $T(x_1+x_2, y_1+y_2) = T(x_1,y_1) + T(x_2,y_2)$ for $(x_1,y_1),\ (x_2,y_2) \in \mathbb{R}^2$.

$$\begin{aligned} T(x_1+x_2, y_1+y_2) \quad &= (2(x_1+x_2)+3(y_1+y_2), -(y_1+y_2), -(x_1+x_2)+2(y_1+y_2)) \\ &\overset{\text{Distributivity}}{=} (2x_1+2x_2+3y_1+3y_2, -y_1-y_2, -x_1-x_2-2y_1-2y_2) \\ &\overset{\text{Associativity}}{=} (2x_1+3y_1+2x_2+3y_2, -y_1-y_2, -x_1-2y_1-x_2-2y_2) \\ &= (2x_1+3y_1, -y_1, -x_1-2y_1) + (2x_2+3y_2, -y_2, -x_2-2y_2) \\ &= T(x_1,y_1) + T(x_2,y_2). \end{aligned}$$

The third property is $T(\lambda x, \lambda y) = \lambda T(x,y)$ for $\lambda \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$:

$$\begin{aligned} T(\lambda x, \lambda y) \quad &= (2\lambda x + 3\lambda y, -\lambda y, -\lambda x + 2\lambda y) \\ &\overset{\text{Distributivity}}{=} \lambda(2x+3y, -y, -x+2y) = \lambda T(x,y). \end{aligned}$$

Hence the transformation T is linear.

(b) Find the matrix corresponding with the linear transformation T. **[0.5]**

Solution: As T is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, the associated matrix has size 3×2 and it is the following one:

$$T = \begin{pmatrix} 2 & 3 \\ 0 & -1 \\ -1 & 2 \end{pmatrix}.$$

(c) Find a basis for the column space of the matrix associated with T. **[0.5]**

Solution: To find a basis for the column space of the matrix T we need a matrix in row echelon form equivalent to T in order to identify the pivot columns:

$$\begin{pmatrix} 2 & 3 \\ 0 & -1 \\ -1 & 2 \end{pmatrix} \underset{R_1/2+R_3 \to R_3}{\sim} \begin{pmatrix} 2 & 3 \\ 0 & -1 \\ 0 & 7/2 \end{pmatrix} \underset{(7/2)R_2+R_3 \to R_3}{\sim} \begin{pmatrix} 2 & 3 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$

The pivot columns are both columns of the matrix T. Thus, the column space of T is spanned by the original columns in T, that is,

$$\operatorname{Col} T = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \right\}.$$

(d) Let $\mathcal{C} = \{c_1, c_2, c_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^3$. The matrix in (b) is given in standard basis $\mathcal{B}$ both in the domaing as in the image of T. Find the matrix associated with the linear transformation T such that the imageis written in the basis $\mathcal{C}$ for $\mathbb{R}^3$. (*Hint: Find a suitable change-of-basis matrix in* $\mathbb{R}^3$.) **[1.25]**

Solution: The change-of-basis matrix from $\mathcal{C}$ to the standard basis $\mathcal{B}$ is given by

$$P_{\mathcal{BC}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix T is written in the corresponding standard basis, that is, $T_{\mathcal{BB}}$. The exercise requires us to find

$$T_{\mathcal{CB}} = P_{\mathcal{CB}} T_{\mathcal{BB}} = (P_{\mathcal{BC}})^{-1} T_{\mathcal{BB}}.$$

Hence we need to compute the inverse matrix of $P_{\mathcal{BC}}$. First compute the determinant:

$$\det(P_{\mathcal{BC}}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \overset{\text{Along 3rd row}}{=} 1(-1)^{3+1} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1.$$

The adjugate matrix of $P_{\mathcal{BC}}$ is:

$$\operatorname{Adj}(P_{\mathcal{BC}}) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

Hence, the inverse matrix of $P_{\mathcal{BC}}$ is:

$$P_{\mathcal{BC}}^{-1} = \frac{1}{\det(P_{\mathcal{BC}})} (\operatorname{Adj}(P_{\mathcal{BC}}))^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Thus, the solution to the exercise is the following matrix:

$$T_{\mathcal{CB}} = P_{\mathcal{CB}} T_{\mathcal{BB}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & 2 \\ 3 & 1 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

(a) Find the eigenvalues of A and their algebraic multiplicities. **[0.75]**

Solution:

To compute the eigenvalues of A we need to find the zeros of the characteristic polynomial:

$$\begin{aligned} P_A(\lambda) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)((1-\lambda)^2-1) = (2-\lambda)(\lambda^2-2\lambda) = -\lambda(\lambda-2)^2. \end{aligned}$$

Then the eigenvalues are 0 and 2. The algebraic multiplicities are $n_0 = 1$ and $n_2 = 2$.

(b) Orthogonally diagonalize the matrix A. Give the matrices P and D. **[1.5]**

Solution: As the matrix is symmetric, it can be orthogonally diagonalized. To find the eigenvectors we must compute the null spaces of $A - \lambda I_3$.

For $\lambda = 0$:

$$A - 0I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \underset{R_2+R_3\to R_3}{\sim} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The homogeneous system with coefficient matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ has one free variable. The system can be rewritten as follows:

$$\begin{aligned} 2a &= 0, & a &= 0 \\ b - c &= 0, & b &= c \\ c &= c. \end{aligned}$$

Thus $\text{Nul}A = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$. The diagonalization is orthogonal if all the eigenvectors are unit. Then we must normalize the vector $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$:

$$\frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{0^2+1^2+1^2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

For $\lambda = 2$:

$$A - 2\mathrm{I}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \underset{-R_2+R_3 \to R_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The homogeneous system with coefficient matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ has two free variables. The system can be rewritten as follows:

$$\begin{aligned} a &= a, \\ -b - c &= 0, \qquad b = -c \\ c &= c. \end{aligned}$$

Thus $\mathrm{Nul}(A - 2\mathrm{I}_3) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$. The diagonalization is orthogonal if all the eigenvectors are unit and orthogonal two by two. Let $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, we compute:

$$\begin{aligned} \|v_2\| &= \sqrt{1^2 + 0^2 + 0^2} = 1, \\ \|v_3\| &= \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}, \\ v_2 \cdot v_3 &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 0 + 0 + 0 = 0. \end{aligned}$$

Then we must normalize the vector v_3:

$$\frac{v_3}{\|v_3\|} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

As the eigenvectors of different eigenvalues are orthogonal among them, we can give now the matrices P and D such that $A = PDP^T$.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Answer the following questions and **always** justify your answers.

(a) If the dimension of the null space of a 4×2 matrix A is 1, what is the rank of A? Why? **[0.5]**

Solution: As A has size 4×2, by the rank theorem we know that $\text{Nul}\, A + \text{rank}\, A = 2$. Then

$$1 + \text{rank}\, A = 2, \quad \text{rank}\, A = 2 - 1 = 1.$$

Thus the rank of A is 1.

(b) Let $A = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$. Find the matrix norm $\|A\|_2$ and describe all the matrices in a singular value decomposition of A. **[0.75]**

Solution: To compute the matrix norm $\|A\|_2$ we must compute the singular values of A. First, compute the eigenvalues of $A^T A$,

$$A^T A = \begin{pmatrix} 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 1 + 0 + 9 = 10.$$

As $A^T A$ is a matrix of size 1×1, it only has one eigenvalue, that is, 10. The singular value of A is $\sigma = \sqrt{10}$, then

$$\|A\|_2 = \max\{\sqrt{10}\} = \sqrt{10}.$$

The singular value decomposition of the matrix A is given by

$$A = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = U\Sigma V^T = U \begin{pmatrix} \sqrt{10} \\ 0 \\ 0 \end{pmatrix} V^T,$$

where U is a 3×3 matrix and V is a 1×1 matrix.

(c) Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $\lambda = 2$ be an eigenvalue of A. Find an **ORTHOGONAL basis** for the eigenspace of eigenvalue 2. **[1]**

Solution: To find an orthogonal basis of the null space of A with eigenvalue $\lambda = 2$, we must find a basis from a matrix in row echelon form equivalent to the following matrix:

$$A - 2\text{I}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{matrix} \sim \\ -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{matrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The homogeneous system to be solved is given by $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The coefficient matrix has two free variables, b and c. The solution set of such a system is given by:

$$\begin{aligned} & a+b+c=0, \quad a=-b-c \\ & b=b \\ & c=c \end{aligned}$$

Thus, $\text{Nul}\,(A-2\text{I}_3) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\}$. Let us see if they are an orthogonal basis:

$$v_1 \cdot v_2 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \neq 0.$$

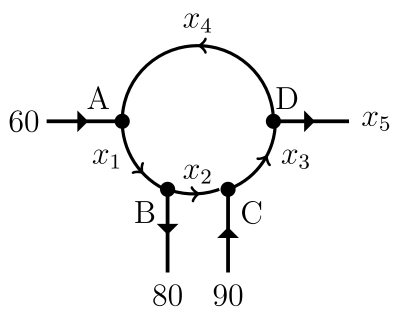
As the scalar product is not zero, the vectors in the basis are not orthogonal and we must use Gram-Schmidt process to obtain an orthogonal basis for $\text{Nul}\,(A-2\text{I}_3)$:

$$\begin{aligned} w_1 &= v_1, \\ w_2 &= v_2 - \text{proj}_{w_1} v_2 = v_2 - \frac{w_1 \cdot v_2}{v_2 \cdot v_2} w_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}. \end{aligned}$$

Note that now w_1 and w_2 are orthogonal since $w_1 \cdot w_2 = \frac{1}{2} - \frac{1}{2} = 0$.

Thus an orthogonal basis for $\text{Nul}(A-2\text{I}_3)$ is given by $\{w_1, w_2\} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} \right\}$.

(d) Consider the following network. Find the system of linear equations that describe the general flow pattern. **Do not solve the system.** **[0.5]**



Solution: The flow of such a diagram is determined by a system of 5 equations with 5 unknowns. There is an equation for each vertex where the flow in must be equal to the flow out. Moreover, the flow arriving into the diagram must be equal to the flow out of the diagram.

$$\begin{array}{lrcl} & \text{Flow in} & = & \text{Flow out} \\ \text{Vertex A} & 60 + x_4 & = & x_1, \\ \text{Vertex B} & x_1 & = & x_2 + 80, \\ \text{Vertex C} & 90 + x_2 & = & x_3, \\ \text{Vertex D} & x_3 & = & x_4 + x_5, \\ \text{Total} & 60 + 90 & = & 80 + x_5. \end{array}$$