Bachelor in Aerospace Engineering

Universidad Carlos III de Madrid
www.uc3m.es

Department of Continuum Mechanics and Structural Engineering

Aerospace Structures

Chapter 4. Laminate and sandwich structures

Composite Beams

Universidad Carlos III de Madrid

Index

Chapter 4. Composite beams and plates

Composite beams

1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Composite beams and plates

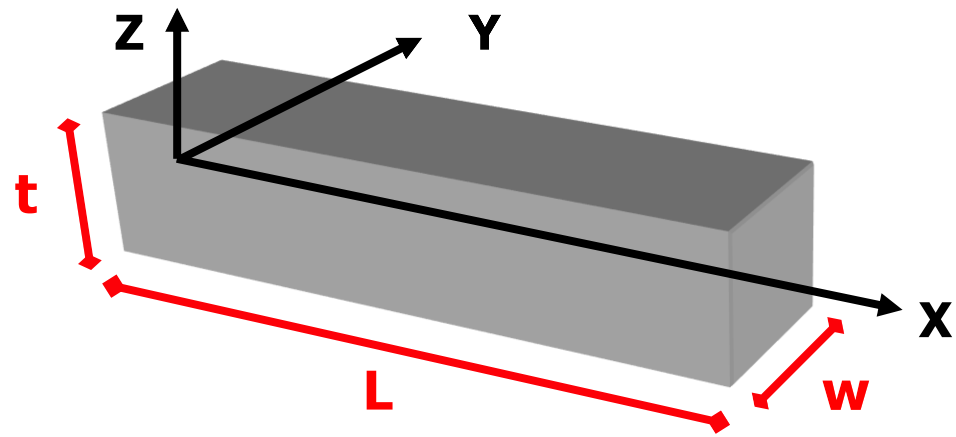
Composite beams

1. **Introduction**
2. Analysis of beams
3. Stresses in solid beams
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Introduction

Previous knowledge



L = Length

t = Thickness

w = Width

- w / L << 1
- t / L << 1
- Loads directions are in plane x-z.
- If loads directions are in plane y-z, we can apply analogous formulation.
- Principle of superposition can be use for loads direction in plane x-z and plane y-z.

Introduction

Internal force in beams:

- Tensile
- Compressive
- Bending moment
- Torsion moment
- Shear
- Combined forces:
  Bending-shear, bending-torsion, …

Manufacture methods

- Pultrusion
- Rolling
- Prepegç
- Resin injection

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Introduction

- Design:
  - Minimize cross-section (weight)
  - Maximize moments of inertia
    - Increase section dimensions
    - Decrease thickness
  - Beams modelled as prismatic elements

- Design stages:

  - Beam dimensions, under quasi-static or dynamic

  - Joint of beam with the rest of the structure

Chapter 4. Composite beams and plates

Composite beams

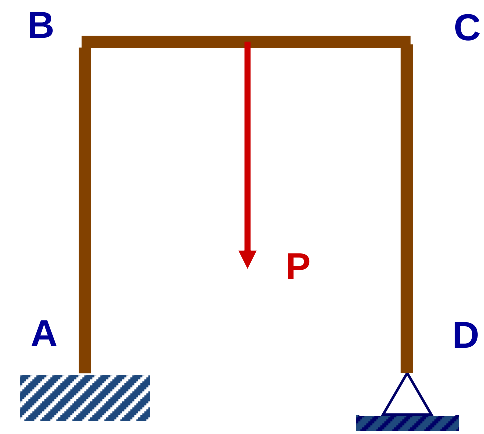
1. Introduction
2. **Analysis of beams**
3. Stresses in solid beams
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

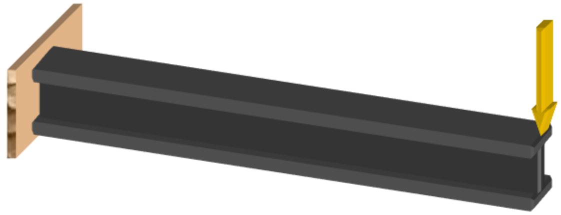
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Calculation process

– Classical beam theory can be used to analyse the structure

- Degree of static indeterminacy
- Reactions calculation
- Internal forces laws





Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Calculation process

– Stresses analysis:

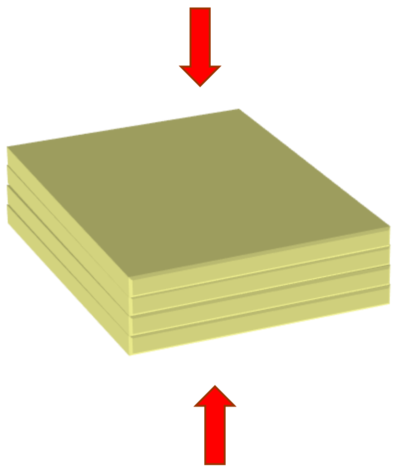
- Laminate theory or first-order-shear-deformation theory can be used to calculate beam strains

- Strains and stresses can be found for each layer

- A failure criterion is applied

Load state

{N} ; {M}



[A] ; [B] ; [D]

Stiffness matrices

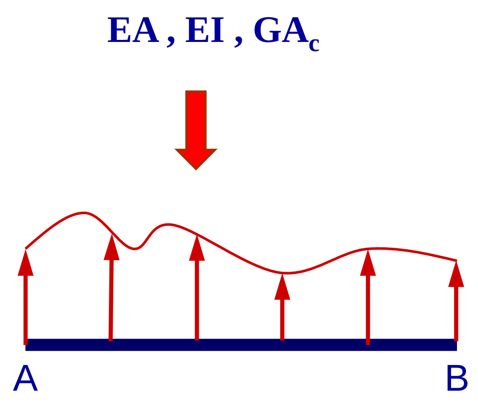
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Calculation process

– Displacement analysis

- Laminate equivalent elastic properties are found using laminate theory or first-order-shear-deformation theory to calculate stiffness matrices

- Navier-Bresse formulation, Mohr theorems or energetic methods can be used to found the displacements



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Composite beams and plates

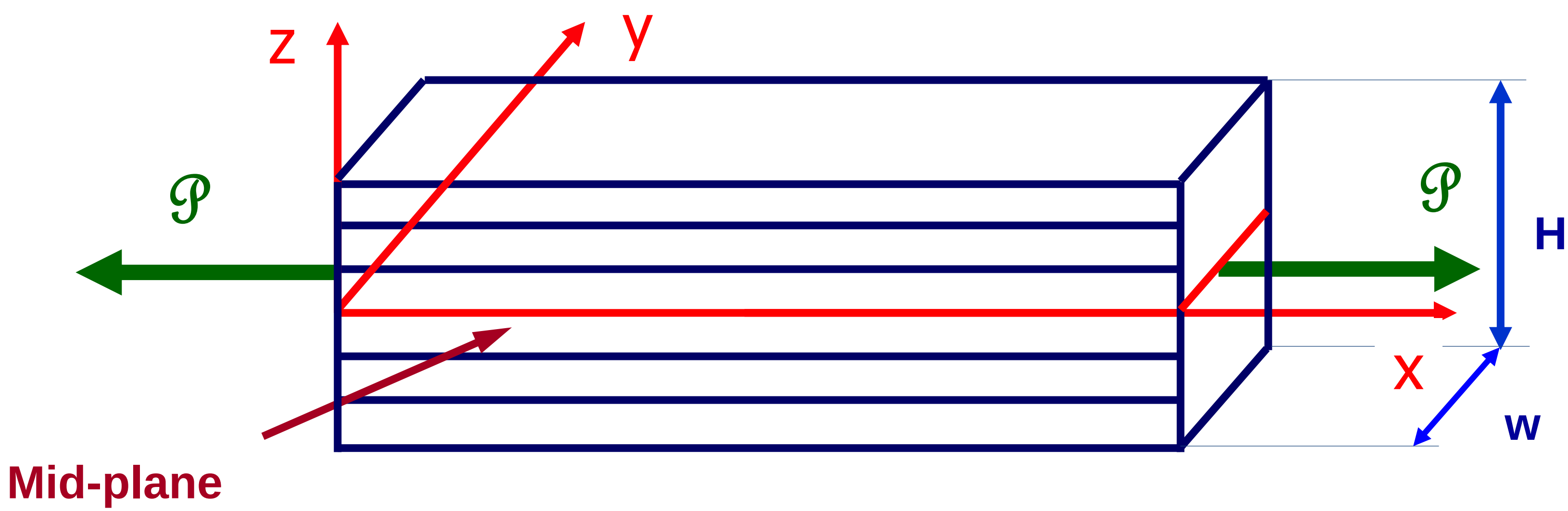
Composite beams

1. Introduction
2. Analysis of beams
3. **Stresses in solid beams**
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Beams under tensile/compressive loads



$\mathcal{P}$: External load (N)

N: Internal axial force per unit length (N/m)

$$N_x = N = \frac{\mathcal{P}}{w}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Beams under tensile/compressive loads

Laminate theory:

$$\{N\} = \underbrace{\left[\int_{-t/2}^{t/2} \left[\overline{Q}\right] \cdot dz \right]}_{[A]} \cdot \left\{ \varepsilon^{0} \right\}$$

$$\{N\} = [A] \cdot \left\{ \varepsilon^{0} \right\} \; in \;\; \mathrm{N/m}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Beams under tensile/compressive loads

Load state:

$$\begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{1S} \\ A_{12} & A_{22} & A_{2S} \\ A_{1S} & A_{2S} & A_{SS} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$$

Strains:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1S} \\ a_{12} & a_{22} & a_{2S} \\ a_{1S} & a_{2S} & a_{SS} \end{bmatrix} \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon_x^0 = a_{11} \cdot N$$

$$N = N(x)$$

$$\varepsilon_x^0 = \varepsilon_x^0(x)$$

One-dimensional problem

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Beams under tensile/compressive loads

Strains of each lamina

Lamina k-th:

$$\left\{\varepsilon\right\}^k = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^k = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \left\{\varepsilon^0\right\}$$

Stresses in each lamina

$$\left\{\sigma\right\}^k = \left[\overline{Q}\right]_k \left\{\varepsilon\right\}^k = \left[\overline{Q}\right]_k \left\{\varepsilon^0\right\}$$

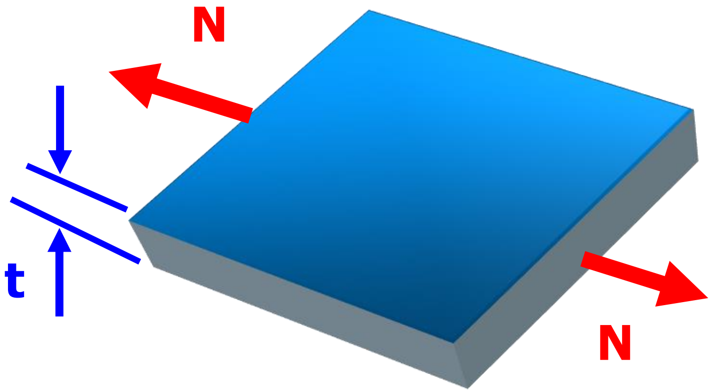
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Beams under tensile/compressive loads

Laminate equivalent constants

$$\varepsilon_x^0 = a_{11} \cdot N \qquad \sigma_x^0 = \frac{N}{t} = \frac{\mathcal{P}}{t \cdot w}$$



$$[a^*] = \frac{1}{[A^*]} \qquad [A^*] = \frac{[A]}{t}$$

In isotropic materials

$$\sigma_x = E \cdot \varepsilon_x \qquad \sigma_x = \frac{\mathcal{P}}{A}$$

$$\Downarrow$$

$$E_x^0 = \frac{1}{a_{11} \cdot t} = \frac{1}{a_{11}^*} \qquad A = w \cdot t$$

$$\boxed{\langle EA \rangle = \frac{w}{a_{11}}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Beams under tensile/compressive loads

Approximate solution

$$\{\varepsilon\}^k = \{\varepsilon^0\} = [\bar{S}]_k \{\sigma\}^k$$

$$\sigma_x^k = E_x^k \, \varepsilon_x^0$$

$$\varepsilon_x^0 = \frac{\sigma_x^k}{E_x^k}$$

$$\sigma_x^k \cdot t_k = E_x^k \cdot t_k \cdot \varepsilon_x^0$$

$$N = \sum_k \sigma_x^k \cdot t_k = \left(\sum_k E_x^k \cdot t_k\right) \cdot \varepsilon_x^0$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Beams under tensile/compressive loads

Where:

$$E_x^0 = \frac{\sum_k E_x^k \cdot t_k}{t}$$

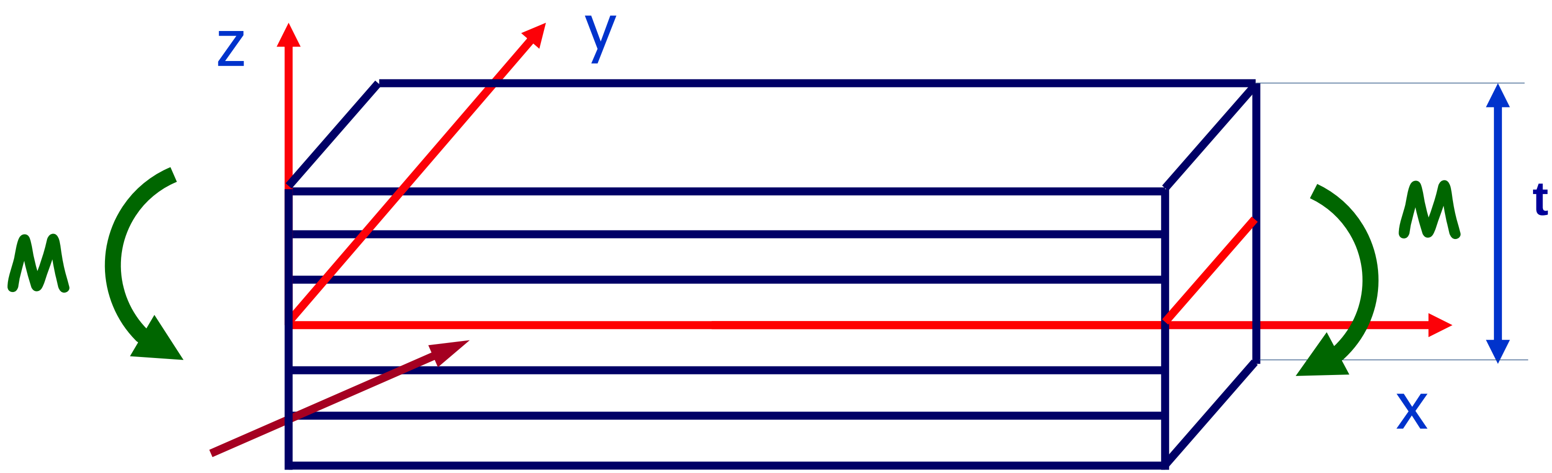
$$N = \sum_k \sigma_x^k \cdot t_k = \left(\sum_k E_x^k \cdot t_k\right) \cdot \varepsilon_x^0 = E_x^0 \cdot t \cdot \varepsilon_x^0$$

$$\sigma_x^0 = \frac{N}{t} = E_x^0 \cdot \varepsilon_x^0$$

$$\varepsilon_x^0 = \frac{1}{E_x^0} \frac{N}{t}$$

Stresses in solid beams

Bending of beams



M: External moment (N·m)

M_x: Bending moment per unit length (N)

$$M_x = M = \frac{\mathcal{M}}{w}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Bending of beams

(Narrow beams)

Load state:

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix}$$

Laminate theory:

Symmetric laminate under pure bending moment

$$\begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{1S} \\ D_{12} & D_{22} & D_{2S} \\ D_{1S} & D_{2S} & D_{SS} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending of beams

Laminate curvatures:

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{1S} \\ d_{12} & d_{22} & d_{2S} \\ d_{1S} & d_{2S} & d_{SS} \end{bmatrix} \begin{Bmatrix} M \\ 0 \\ 0 \end{Bmatrix}$$

$$K_x = d_{11} \cdot M$$

$$M = M(x)$$

$$k_x = k_x(x)$$

One-dimensional problem

Bending of beams

Strains in each lamina

Lamina k-th:

$$\{\varepsilon\}^k = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^k = z \cdot \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = z \cdot \{k\}$$

Stresses in each lamina

$$\{\sigma\}^k = [\overline{Q}]_k \{\varepsilon\}^k$$

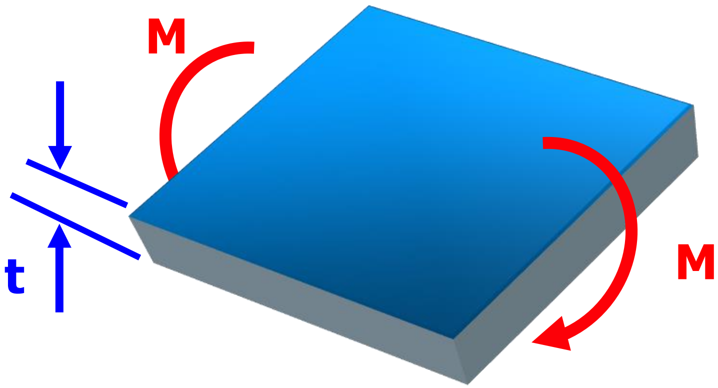
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Bending of beams

Laminate equivalent constants

$$\kappa_x = d_{11} \cdot M = \frac{1}{\rho_x}$$



In isotropic materials: $\frac{1}{\rho_x} = \frac{M}{E \cdot I}$

$$\Downarrow$$

$$I_y = \frac{1}{12} \cdot w \cdot t^3$$

$$E_x^b = \frac{12}{t^3 \cdot d_{11}} = \frac{1}{d_{11}^*}$$

$$\boxed{\langle EI \rangle = \frac{w}{d_{11}}}$$

$$[d^*] = \frac{1}{[D^*]} \qquad [D^*] = \frac{12 \cdot [D]}{t^3}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending of beams

Laminate average maximum stresses

In symmetrical laminates under bending an equivalent maximum stresses can be defined as:

$$\sigma_x^b = \frac{6M}{t^2} \quad \sigma_y^b = 0 \quad \tau_x^b = 0$$

$$\{\sigma\}^b = \left\{ \begin{array}{c} \dfrac{6M}{t^2} \\ 0 \\ 0 \end{array} \right\} = \frac{6}{t^2}\{M\}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Bending of beams

Laminate maximum strains

Equivalent maximum strains:

$$\{\varepsilon\}^b = \frac{t}{2}\cdot\{\kappa\}$$

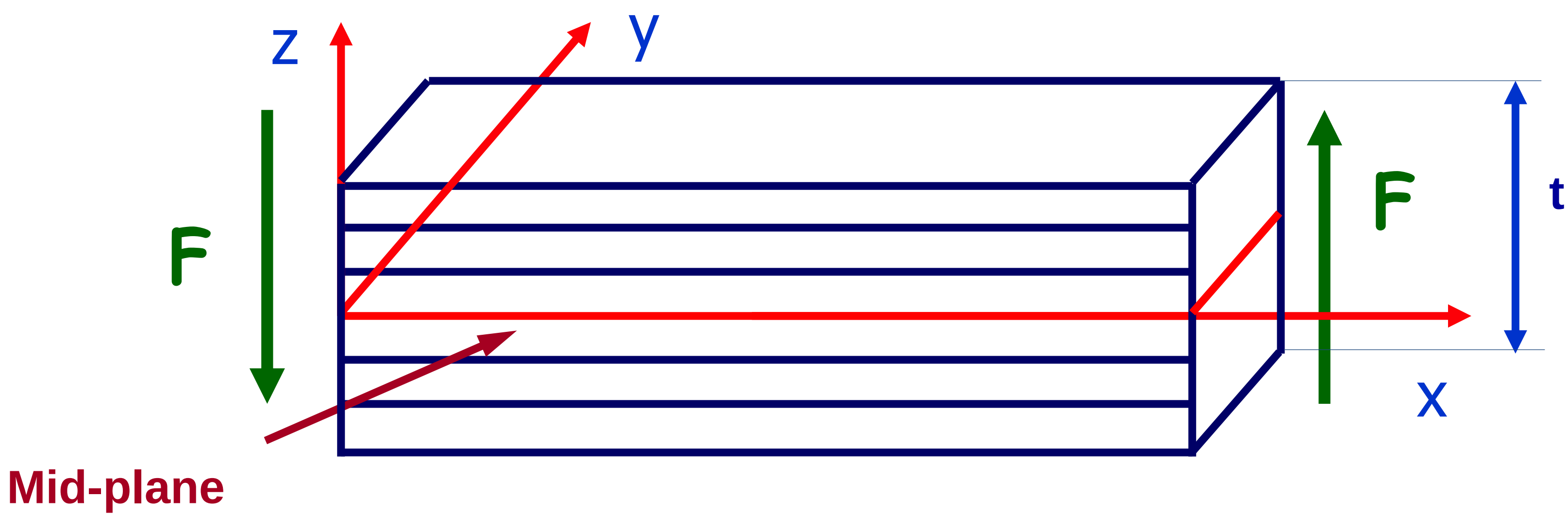
$$\{\varepsilon\}^b = \frac{t}{2}\cdot\{\kappa\} = \frac{t}{2}[D]^{-1}\{M\} \Rightarrow \left\{\begin{aligned} \varepsilon_x^b &= \frac{t}{2} d_{11} M \\ \varepsilon_y^b &= \frac{t}{2} d_{12} M \\ \gamma_{xy}^b &= \frac{t}{2} d_{1S} M \end{aligned}\right.$$

Since a linear through-the-thickness strains variation is assumed, strains at any layer can be found. Therefore stresses can be also found.

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in solid beams

Interlaminar shear forces in beams



F: External force (N)

N_{xy}: Shear force per unit length (N/m)

$$Q_x = Q = \frac{F}{w}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Interlaminar shear forces in beams

Load state:

$$\{Q\} = \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q \end{Bmatrix}$$

First-Order Shear Deformation theory:

$$\begin{Bmatrix} 0 \\ Q \end{Bmatrix} = K \begin{Bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{Bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

Strains:

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \frac{1}{K} \begin{Bmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{Bmatrix} \begin{Bmatrix} 0 \\ Q \end{Bmatrix}$$

$$\gamma_{xz} = \frac{a_{55}}{K} \cdot Q$$

One-Dimensional problem

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Interlaminar shear forces in beams

Lamina strains

Lamina k-th:

$$\{\gamma\}^k = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^k = \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \{\gamma^0\}$$

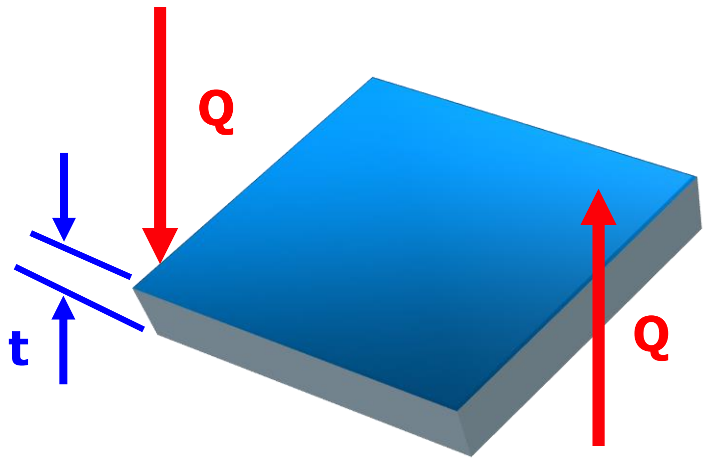
Lamina stress:

$$\{\tau\}^k = [\bar{Q}]_k \cdot \{\gamma\}^k = [\bar{Q}]_k \cdot \{\gamma^0\}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Interlaminar shear forces in beams

Laminate equivalent constants



$$\gamma_{xz}^{0} = \frac{a_{55}}{K} \cdot Q \qquad \tau_{xz}^{o} = \frac{Q}{t}$$

In isotropic materials:

$$\tau_{xz} = G \cdot \gamma_{xz} \qquad \tau_{xz} = \frac{F}{A_s}$$

$$\Downarrow$$

$$G_{xz}^{0} = \frac{1}{a_{55} \cdot t} \qquad A_s = K \cdot w \cdot t$$

$$\boxed{\langle GA_s \rangle = \frac{K \cdot w}{a_{55}}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Composite beams and plates

Composite beams

1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. **Displacements in solid beams**
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

- **Long beams**

  – L/H>10

  – The shear force contribution to the displacements can be neglected. Displacements are mainly produced by bending moment

$$d = d_b$$

  d_b controlled by EI

- **Short beams**

  – L/H<10

  – The shear force contribution to the displacements can <u>not</u> be neglected

$$d = d_b + d_s$$

  d_b controlled by EI
  d_s controlled by GA

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Calculation process

- Classical laminate theory is used to find equivalent stiffness to bending moments (E_b) and shear forces (G_{xz})
- Navier-Bresse formulation can be used to find the displacements

$$\theta_B = \theta_A - \int_A^B \frac{M}{EI} \cdot ds$$

$$v_B = v_A + \theta_A \cdot (z_B - z_A) + \int_A^B \left(\frac{N}{EA} dy + \frac{Q}{GA_s} dZ \right) - \int_A^B \frac{M}{EI} (z_B - z) \cdot ds$$

$$w_B = w_A + \theta_A \cdot (y_B - y_A) + \int_A^B \left(\frac{N}{EA} dz - \frac{Q}{GA_s} dy \right) + \int_A^B \frac{M}{EI} (y_B - y) \cdot ds$$

- Where

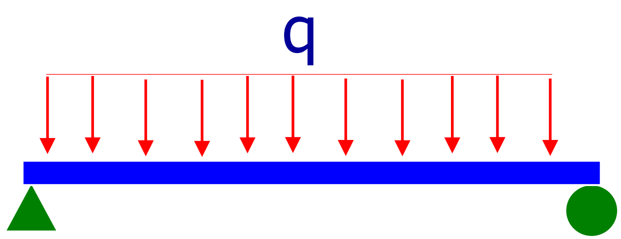
$E \rightarrow Ex_b$

$G \rightarrow G_{xz}$

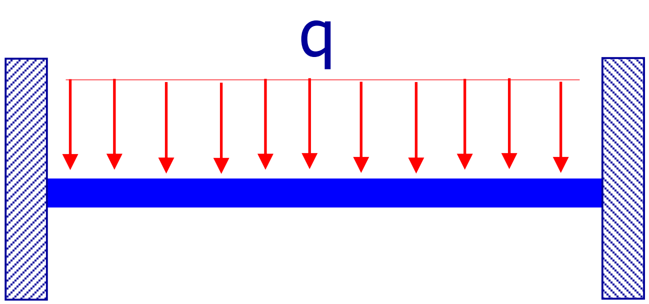
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Maximum displacements



$$\delta_{\max} = \frac{5}{384}\frac{qL^4}{\langle EI \rangle} + \cancel{\frac{1}{8}\frac{qL^2}{\langle GA_s \rangle}}$$

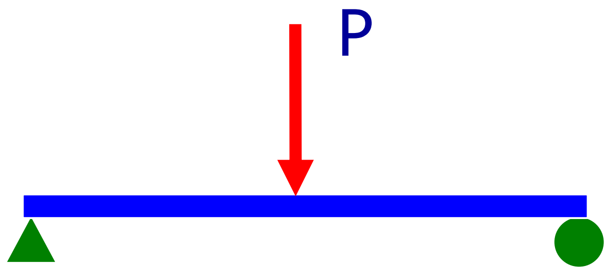


$$\delta_{\max} = \frac{1}{384}\frac{qL^4}{\langle EI \rangle} + \cancel{\frac{1}{8}\frac{qL^2}{\langle GA_s \rangle}}$$

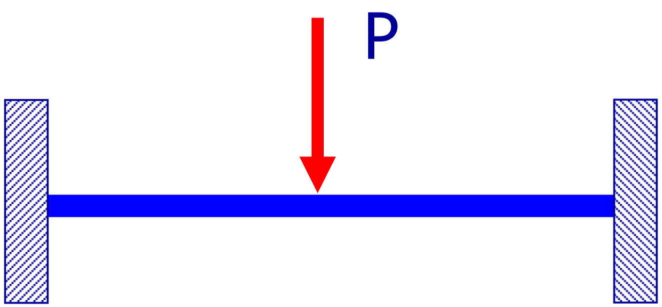
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Maximum displacements



$$\delta_{\max} = \frac{1}{48}\frac{PL^3}{\langle EI \rangle} + \cancel{\frac{1}{4}\frac{PL}{\langle GA_s \rangle}}$$



$$\delta_{\max} = \frac{1}{192}\frac{PL^3}{\langle EI \rangle} + \cancel{\frac{1}{4}\frac{PL}{\langle GA_s \rangle}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Maximum displacements

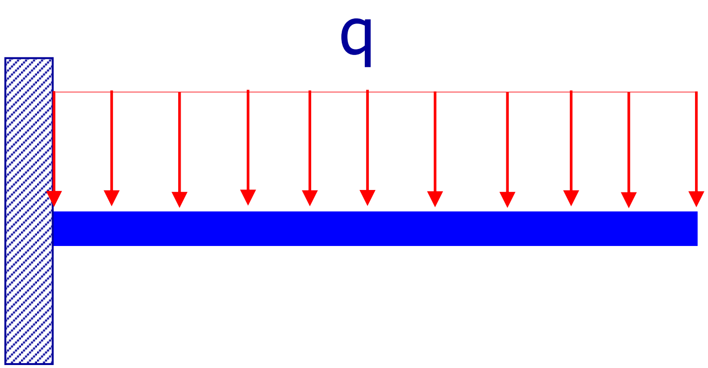
$$\delta_{\max} = \frac{ML^2}{8\langle EI \rangle}$$

$$\delta_{\max} = \frac{PL^3}{3\langle EI \rangle} + \cancel{\frac{PL}{\langle GA_s \rangle}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Maximum displacements

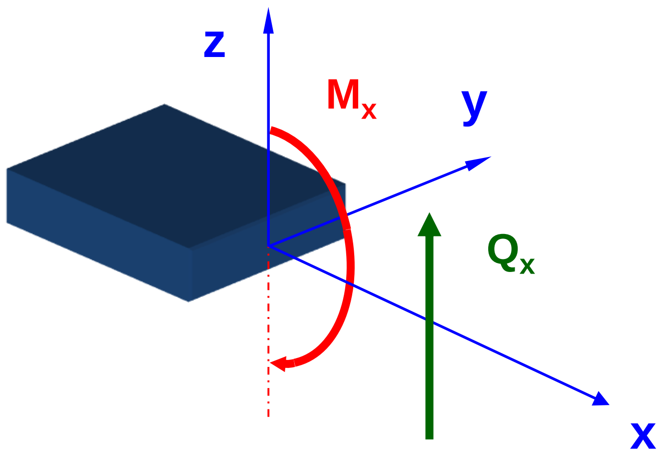


$$\delta_{\max} = \frac{1}{8}\frac{qL^4}{\langle EI \rangle} + \cancel{\frac{1}{2}\frac{qL^2}{\langle GA_s \rangle}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Analysis of beams



$M_y = 0$

$M_{xy}=0$

$Q_y=0$

$\phi_y=0$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Analysis of beams

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{1s} \\ D_{12} & D_{22} & D_{2s} \\ D_{1s} & D_{2s} & D_{ss} \end{bmatrix} \cdot \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

$$\{k\} = [d]\{M\} \Leftrightarrow$$

$$k_x = \frac{\partial \phi_x}{\partial x}$$

$$k_y = \frac{\partial \phi_y}{\partial y}$$

$$k_{xy} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Analysis of beams

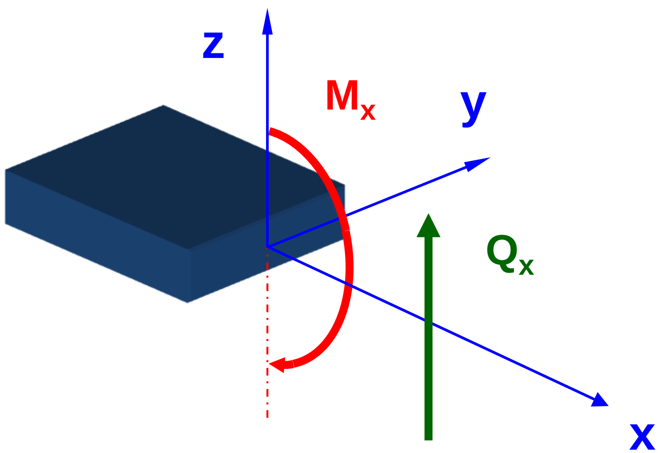
$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \cdot \begin{Bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{Bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

$$\{\gamma\} = \frac{1}{K}[a]\{Q\} \quad \Leftrightarrow \quad \begin{aligned} \gamma_{yz} &= \frac{\partial w}{\partial y} + \phi_y \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \phi_x \end{aligned}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in solid beams

Analysis of beams



$$\frac{\partial \phi_x}{\partial x} = d_{11} M_x$$

$$\frac{\partial w}{\partial x} + \phi_x = \frac{a_{55}}{K} Q_x$$

$M_y = 0$ $Q_y = 0$

$M_{xy} = 0$ $\phi_y = 0$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Composite beams and plates

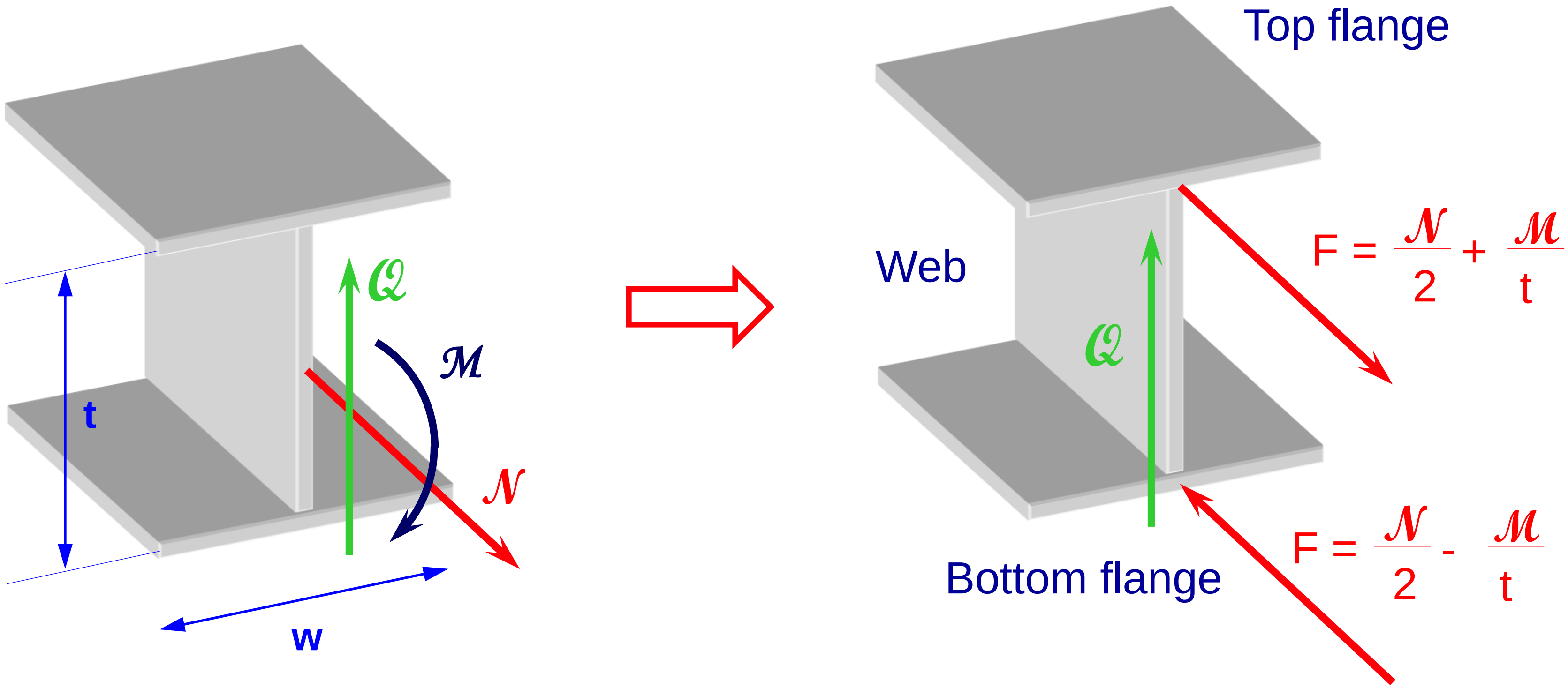
Composite beams

1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. Displacements in solid beams
5. **Stresses in thin-walled beams**
6. Displacements in thin-walled beams
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in thin-walled beams

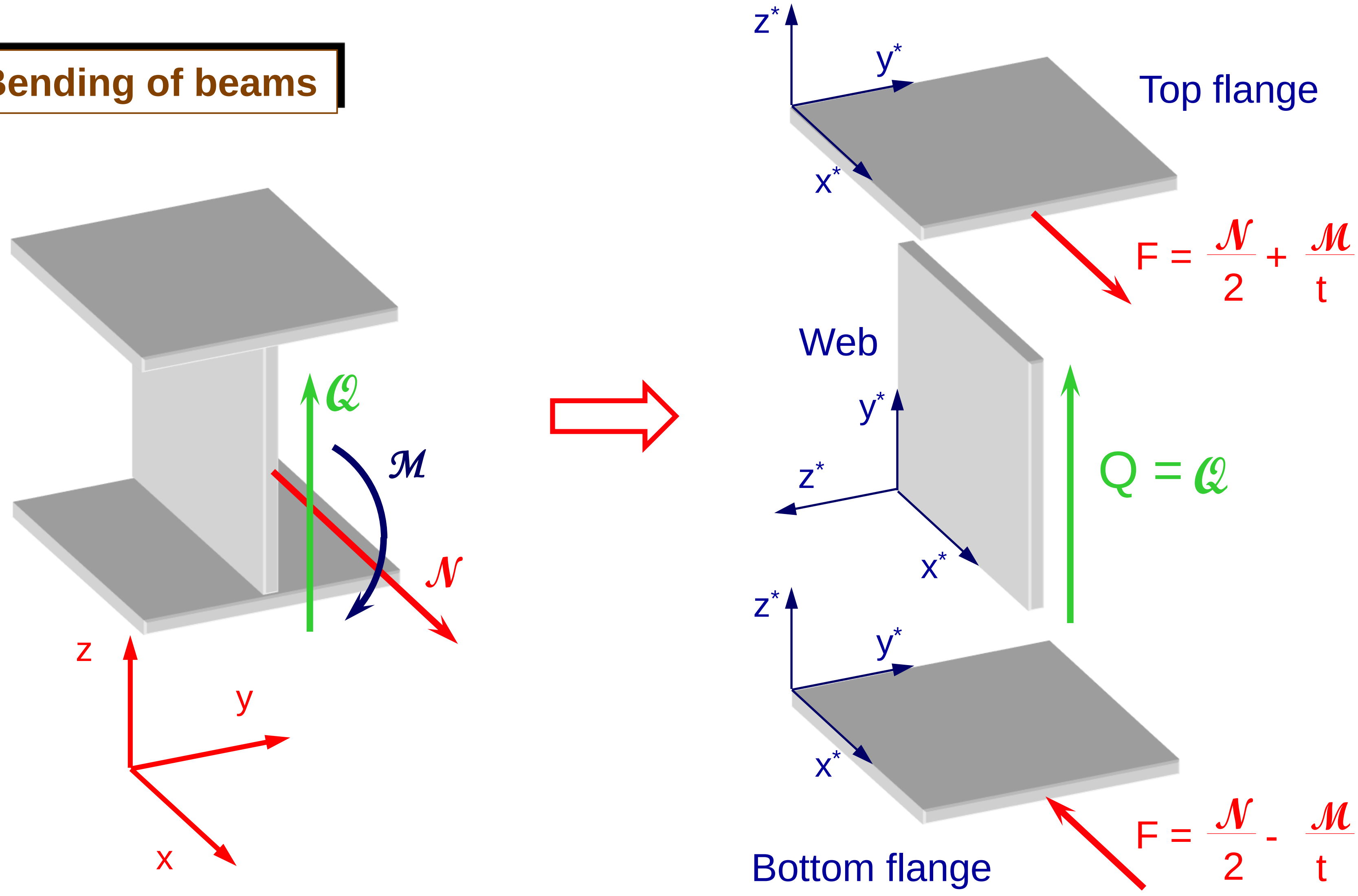
Bending of beams



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in thin-walled beams

Bending of beams



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in thin-walled beams

Bending of beams

Top flange

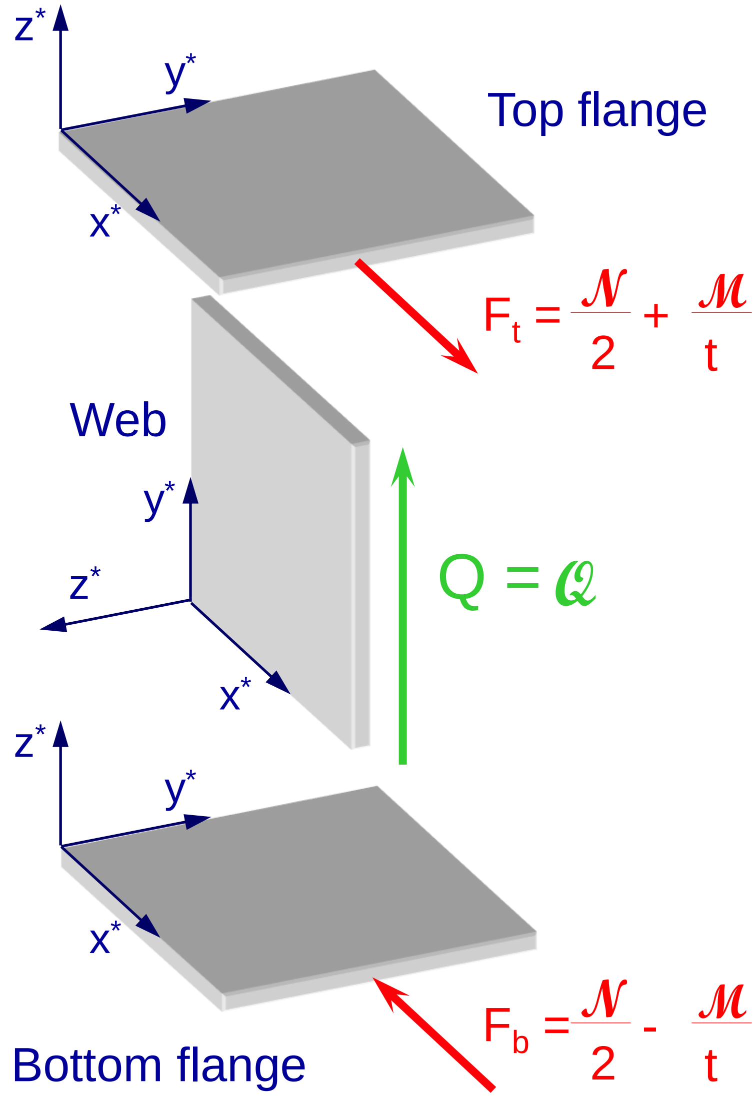
$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} F_t/w \\ 0 \\ 0 \end{Bmatrix}$$

Web

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q/t \end{Bmatrix}$$

Bottom flange

$$\{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} F_b/w \\ 0 \\ 0 \end{Bmatrix}$$



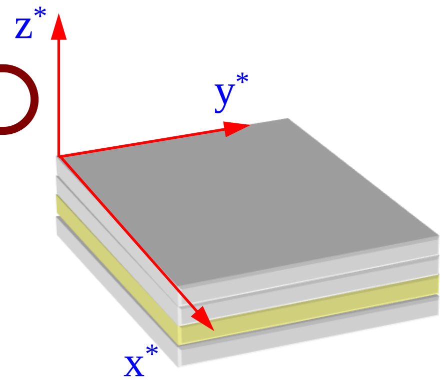
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Stresses in thin-walled beams

Bending of beams

Load state

$\{N\}$



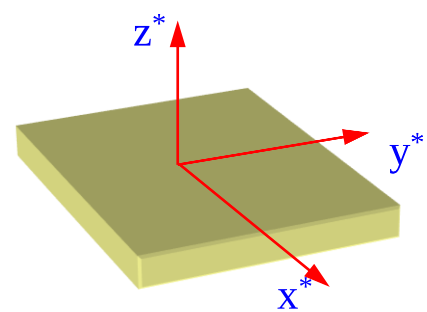
Strains

$$\{N\} = [A] \cdot \{\varepsilon^0\}$$

Stresses field

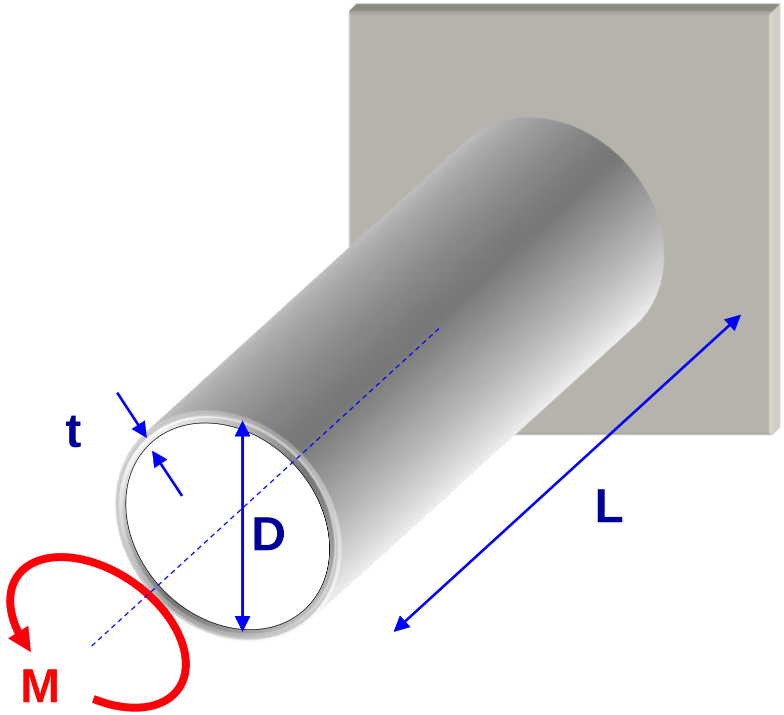
(In each layer)

$$\{\sigma\}^{xy}{}_i = [\overline{Q}]_i \cdot \left(\{\varepsilon^0\} + z \cdot \{k\}\right)$$



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Torsion of beams



Hypothesis:

H<<D

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Torsion of beams

Internal forces

Theory for isotropic materials:

$$\tau_{xy}^{0} = \frac{2 \cdot M}{\pi \cdot D^2 \cdot t} \quad \Rightarrow \quad \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{2 \cdot M}{D \cdot t} \end{Bmatrix}$$

Torsion of beams

Strains

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_x^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1s} \\ a_{12} & a_{22} & a_{22} \\ a_{1s} & a_{22} & a_{ss} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dfrac{2 \cdot M}{D \cdot t} \end{Bmatrix}$$

En particular $$\gamma_{xy}^o = a_{ss} \cdot \frac{2 \cdot M}{D \cdot t}$$

Stresses in thin-walled beams

Torsion of beams

Strains

$$\gamma_{xy}^{o} = a_{ss} \cdot \frac{2 \cdot M}{D \cdot t}$$

$$\gamma_{xy}^{\ 0} = \frac{2 \cdot M}{\pi \cdot D^2 \cdot t \cdot G_{xy}^{0}} \qquad G_{xy}^{0} = \frac{\pi \cdot D}{a_{ss}}$$

Rotations:

$$\theta_{\max} = \frac{M \cdot L}{G_{xy}^{0} \cdot I_o} \qquad I_o = \frac{\pi \cdot D^3 \cdot t}{4}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Composite beams and plates

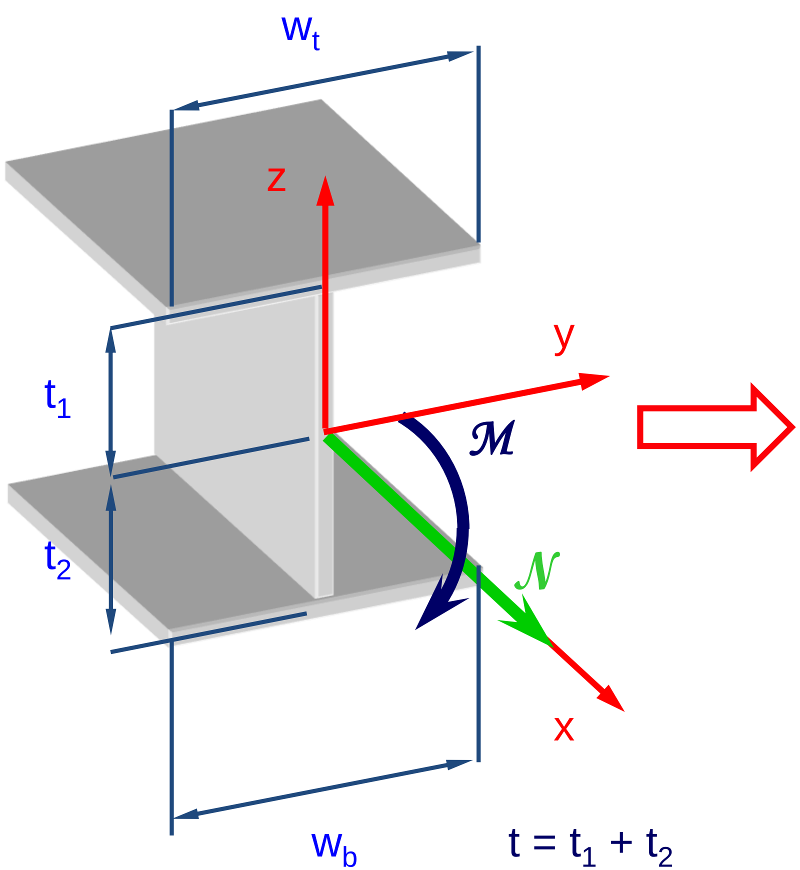
Composite beams

1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. **Displacements in thin-walled beams**
7. References

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment



Hypotheses

- The bending moment can be found through the integral of the normal stress along the z direction.

- Longitudinal strain in x direction is directly related to the distance from neutral axis.

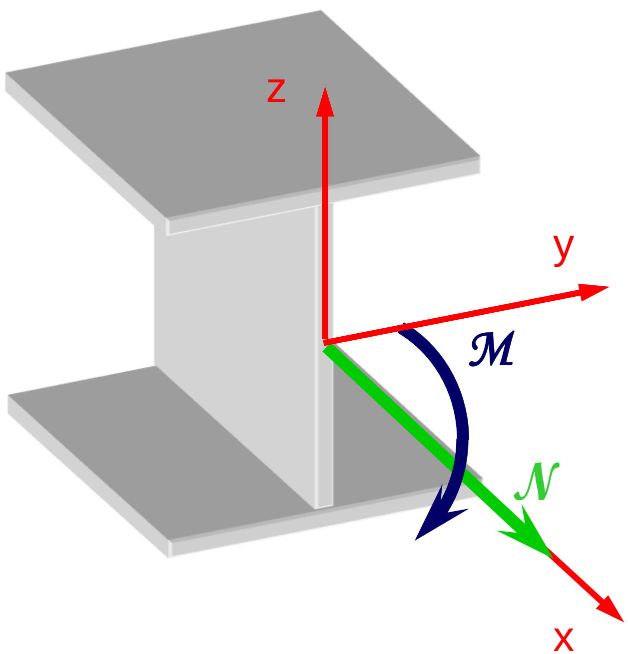
$$\varepsilon_x(z) = z \cdot k_x$$

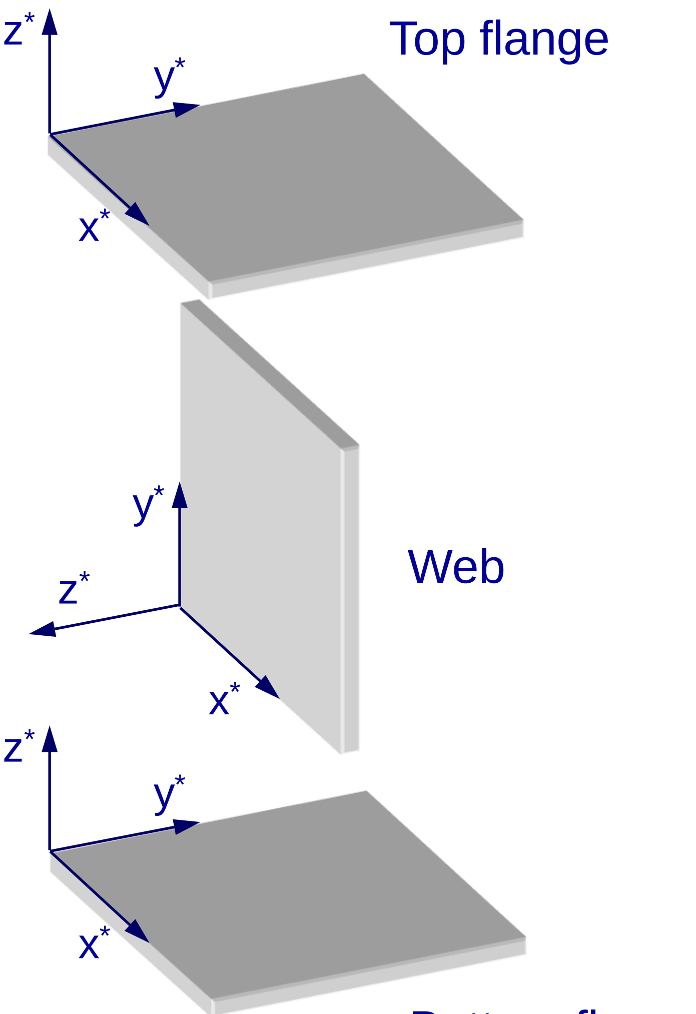
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Universidad Carlos III de Madrid

Displacements in thin-walled beams

Axial force and bending moment





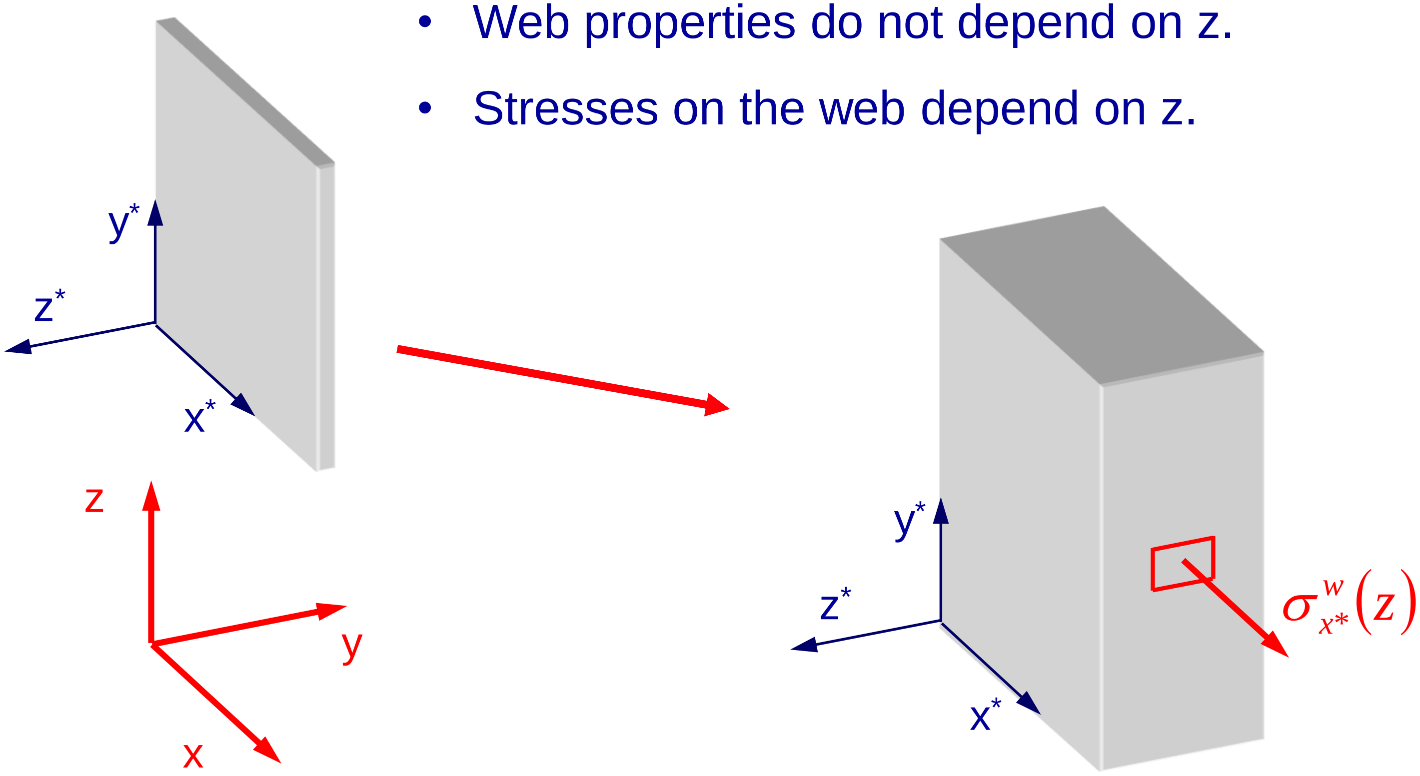
Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment

- Bending stiffness matrix [D] is not necessary.
- Web properties do not depend on z.
- Stresses on the web depend on z.

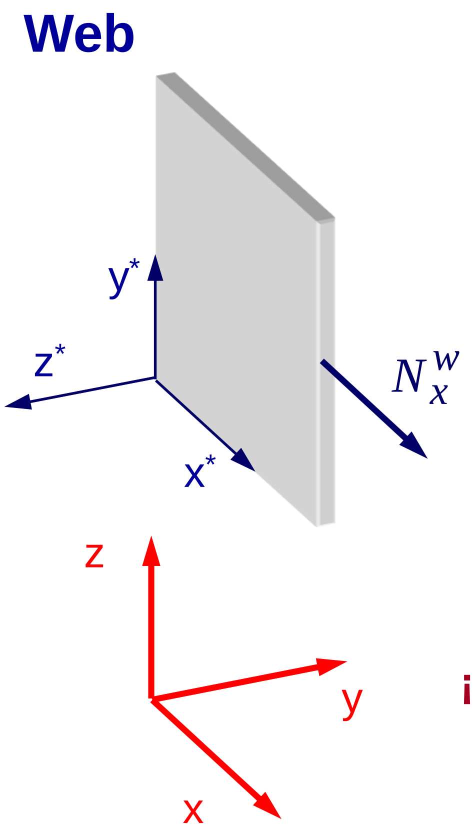
Web



Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment



$$N_{x*}^{w} = \int \sigma_{x*}^{w} \cdot dz*$$

$$\begin{Bmatrix} \varepsilon_{x*}^{o\,w} \\ \varepsilon_{y*}^{o\,w} \\ \gamma_{xy*}^{o\,w} \end{Bmatrix} = \begin{bmatrix} a_{11*}^{w} & a_{11*}^{w} & a_{11*}^{w} \\ a_{11*}^{w} & a_{11*}^{w} & a_{11*}^{w} \\ a_{11*}^{w} & a_{11*}^{w} & a_{11*}^{w} \end{bmatrix} \cdot \begin{Bmatrix} N_{x*}^{w} \\ 0 \\ 0 \end{Bmatrix}$$

$$\boxed{\varepsilon_{x*}^{o\,w} = a_{11*}^{w} \cdot N_{x*}^{w}}$$

$$\Downarrow$$

$$\varepsilon_{x}^{o\,w}(z) = a_{11*}^{w} \cdot N_{x}^{w}(z)$$

¡ Axes x* and x match up !

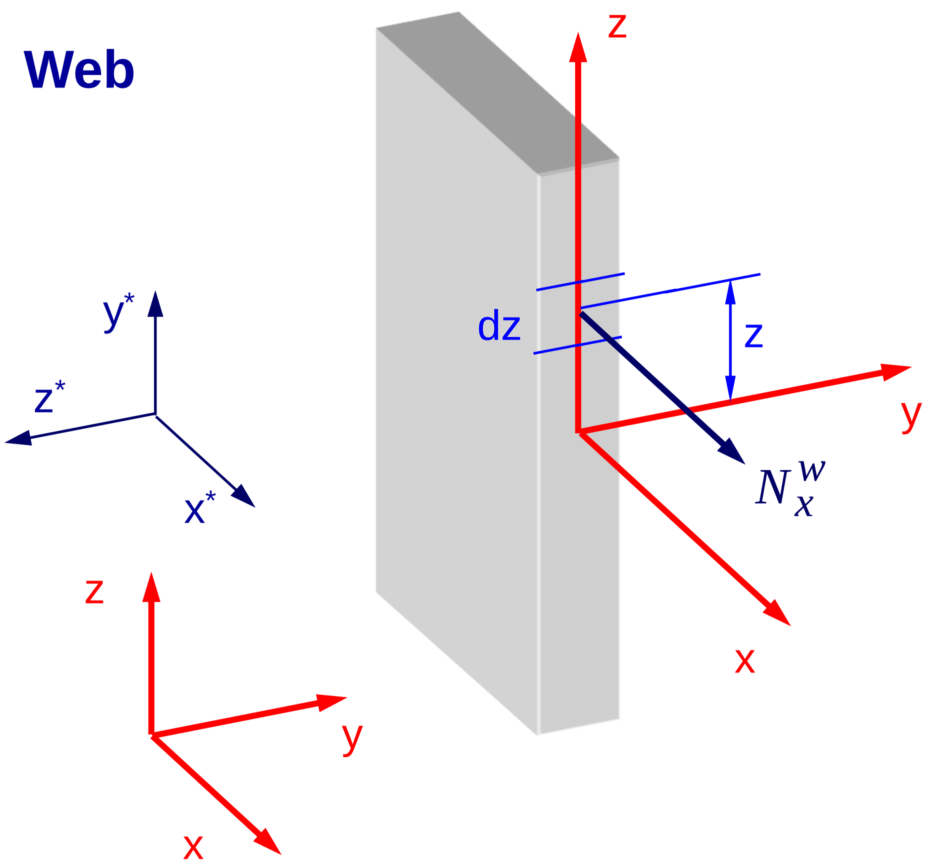
¡ Results depend on global coordinate z!

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment

Web



$$\varepsilon_x^{o^w}(z) = z \cdot k_x$$

$$N_x^w = \frac{1}{a_{11*}^w} \cdot z \cdot k_x$$

Axial force per unit length, coordinate y* (coordinate z)

This axial force contributes to the section bending moment

$$\mathcal{M}_w = \int_{-t/2}^{t/2} N_x^w \cdot z \cdot dz$$

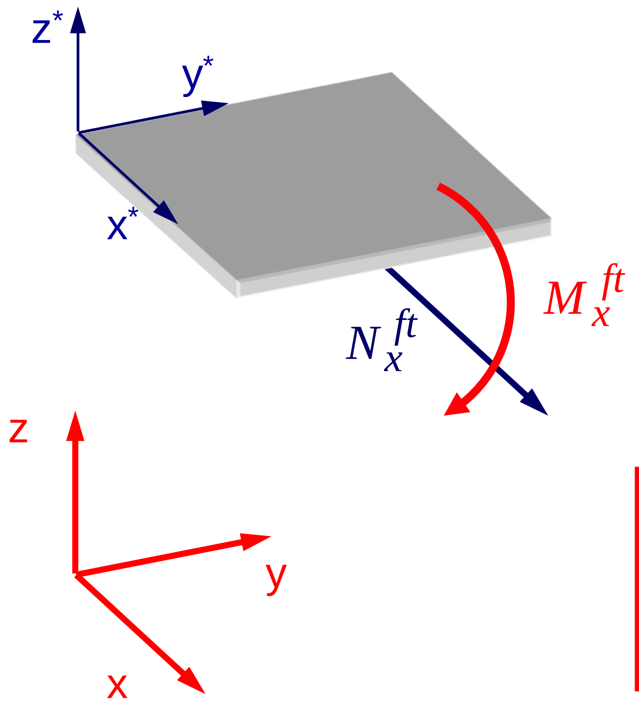
$$\boxed{\mathcal{M}_w = \frac{t^3}{12 \cdot a_{11*}^w} \cdot k_x}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment

Top flange



$$\begin{Bmatrix} \varepsilon_{x^*}^{o\,ft} \\ \varepsilon_{y^*}^{o\,ft} \\ \gamma_{xy^*}^{o\,ft} \end{Bmatrix} = \begin{bmatrix} a_{11^*}^{ft} & a_{11^*}^{ft} & a_{11^*}^{ft} \\ a_{11^*}^{ft} & a_{11^*}^{ft} & a_{11^*}^{ft} \\ a_{11^*}^{ft} & a_{11^*}^{ft} & a_{11^*}^{ft} \end{bmatrix} \cdot \begin{Bmatrix} N_{x^*}^{ft} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} k_{x^*} \\ k_{y^*} \\ k_{xy^*} \end{Bmatrix} = \begin{bmatrix} d_{11^*}^{ft} & d_{11^*}^{ft} & d_{11^*}^{ft} \\ d_{11^*}^{ft} & d_{11^*}^{ft} & d_{11^*}^{ft} \\ d_{11^*}^{ft} & d_{11^*}^{ft} & d_{11^*}^{ft} \end{bmatrix} \cdot \begin{Bmatrix} M_{x^*}^{ft} \\ 0 \\ 0 \end{Bmatrix}$$

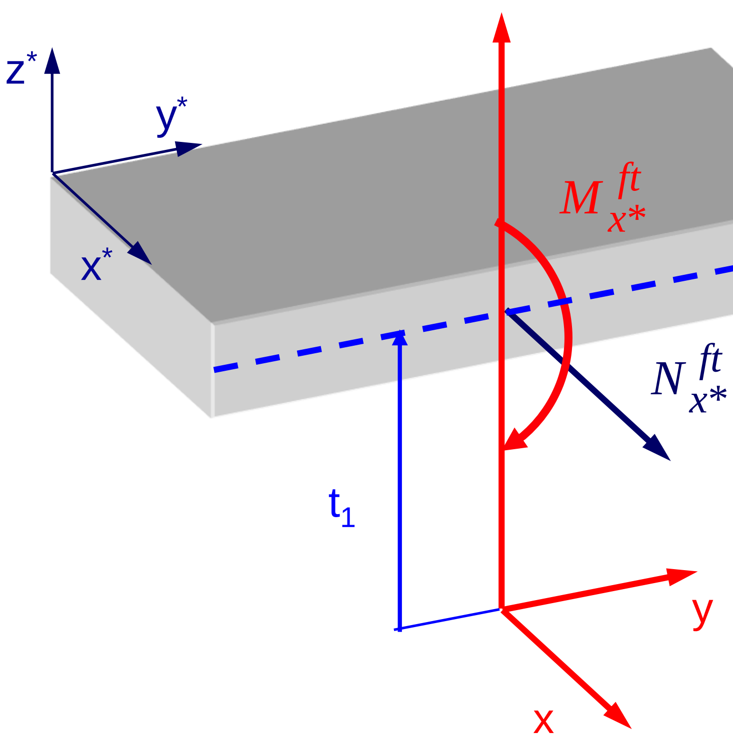
$$\boxed{\begin{aligned} \varepsilon_{x^*}^{o\,ft} &= a_{11^*}^{ft} \cdot N_{x^*}^{ft} \\ k_{x^*} &= d_{11^*}^{ft} \cdot M_{x^*}^{ft} \end{aligned}} \;\Rightarrow\; \begin{aligned} \varepsilon_{x}^{o\,ft} &= a_{11^*}^{ft} \cdot N_{x}^{ft} \\ k_{x} &= d_{11^*}^{ft} \cdot M_{x}^{ft} \end{aligned}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment

Top flange



Axial force and bending moment in the flange contribute to the section bending moment

$$\mathcal{M}_{ft} = w_t \cdot \left(N_x^{ft} \cdot t_1 + M_x^{ft} \right)$$

$$\mathcal{M}_{ft} = w_t \cdot \left(\frac{t_1}{a_{11*}^{ft}} \cdot \varepsilon_x^{o\,ft} + \frac{1}{d_{11*}^{ft}} \cdot k_x \right)$$

Strains at flange mid-plane

$$\varepsilon_x^{o\,ft} = t_1 \cdot k_x$$

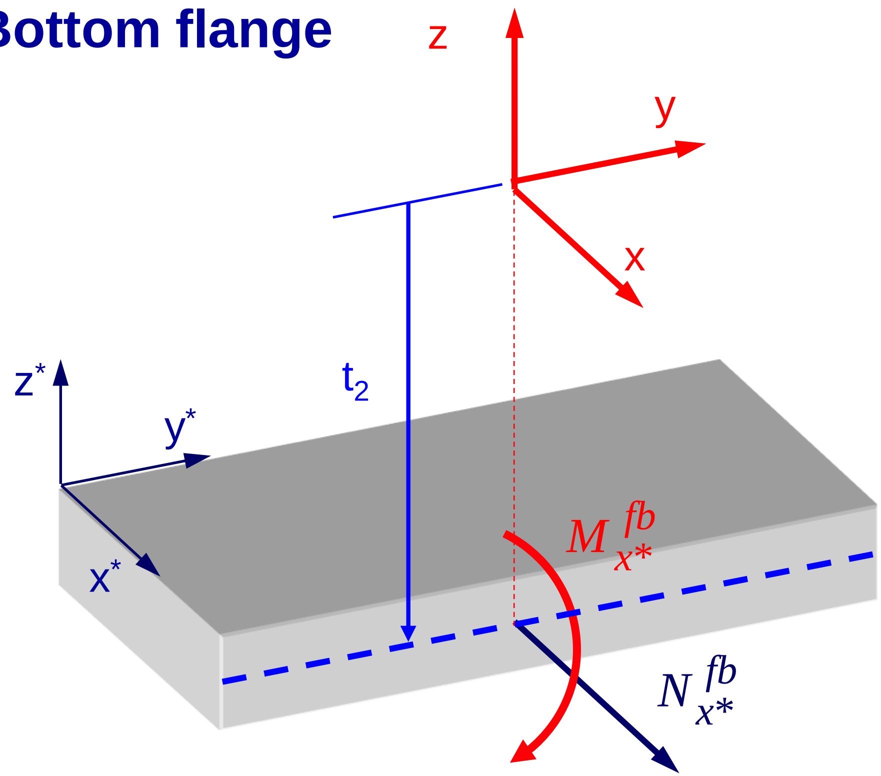
$$\boxed{\mathcal{M}_{ft} = w_t \cdot \left(\frac{t_1^2}{a_{11*}^{ft}} \cdot + \frac{1}{d_{11*}^{ft}} \right) \cdot k_x}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment

Bottom flange



By analogy:

$$\boxed{\mathcal{M}_{fb} = w_b \cdot \left(\frac{t_2^2}{a_{11*}^{fb}} \cdot + \frac{1}{d_{11*}^{fb}} \right) \cdot k_x}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Displacements in thin-walled beams

Axial force and bending moment

The section bending moment is the sum of the bending moments supported by web and flanges:

$$\mathcal{M} = \left[w_t \cdot \left(\frac{t_1^2}{a_{11*}^{ft}} \cdot + \frac{1}{d_{11*}^{ft}} \right) + w_b \cdot \left(\frac{t_2^2}{a_{11*}^{fb}} \cdot + \frac{1}{d_{11*}^{fb}} \right) + \frac{(t_1 + t_2)^3}{12 \cdot a_{11*}^{w}} \right] \cdot k_x$$

In isotropic material beams:

$$\mathcal{M} = EI \cdot k_x$$

$$\boxed{\langle EI \rangle = w_t \cdot \left(\frac{t_1^2}{a_{11*}^{ft}} \cdot + \frac{1}{d_{11*}^{ft}} \right) + w_b \cdot \left(\frac{t_2^2}{a_{11*}^{fb}} \cdot + \frac{1}{d_{11*}^{fb}} \right) + \frac{(t_1 + t_2)^3}{12 \cdot a_{11*}^{w}}}$$

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

Chapter 4. Composite beams and plates

Composite beams

1. Introduction
2. Analysis of beams
3. Stresses in solid beams
4. Displacements in solid beams
5. Stresses in thin-walled beams
6. Displacements in thin-walled beams
7. **References**

Authors: Enrique Barbero Pozuelo, José Fernández Sáez, Carlos Santiuste Romero

References

The Behaviour of Structures Composed of Composite Materials

J.R. Vinson y R.L. Sierakowski

Kluwer Academic Publishers. 2002.

Cap. 4: Beams, columns and Rods of Composite Materials

Mechanics of laminated composite plates. Theory and analysis

J.N. Reddy

CRC Press. 1997.

Cap 6: One- Dimensional Analysis of Laminated Plates