UNIVERSIDAD CARLOS III DE MADRID

Escuela Politécnica Superior

Departamento de Matemáticas

PROBLEMS, CALCULUS I, 1st COURSE

1. FUNCTIONS OF A REAL VARIABLE

BACHELOR IN:
Audiovisual System Engineering
Communication System Engineering
Telematics Engineering

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1 Functions of a real variable

1.1 The real line

Problem 1.1.1

i) Consider the real numbers $0 < a < b$, $k > 0$. Prove the inequalities

$$1)\quad a < \sqrt{ab} < \frac{a+b}{2} < b, \qquad 2)\quad \frac{a}{b} < \frac{a+k}{b+k}.$$

ii) Prove that $|a+b| = |a| + |b| \iff ab \geq 0$.

iii) Prove the inequality $|a-b| \geq \Big||a| - |b|\Big|$, for all $a, b \in I\!R$.

iv) Prove that:

$$a)\ \max\{x,y\} = \frac{x+y+|x-y|}{2}, \qquad b)\ \min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

v) Express in a single formula the following function $f(x) = (x)_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$.

Problem 1.1.2 Decompose the expressions in n as a product of factors to prove that, for all $n \in I\!N$ we have

i) $n^2 - n$ is even;

ii) $n^3 - n$ is a multiple of 6;

iii) $n^2 - 1$ is a multiple of 6 if n is even.

Problem 1.1.3 Use the induction technique to prove the following formulas:

$$i)\ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}; \qquad ii)\ \sum_{j=1}^{n} (2j-1) = n^2; \qquad iii)\ \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 1.1.4 Prove by induction

i) Geometrical sum: $\displaystyle\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$, for all $n \in I\!N$, $r \neq 1$;

ii) Bernouilli inequality: $(1+h)^n \geq 1 + nh$, for all $n \in I\!N$, $h > -1$.

Problem 1.1.5 Prove by induction that for all $n \in I\!N$ we have

i) $10^n - 1$ is a multiple of 9;

ii) $10^n - (-1)^n$ is a multiple of 11.

Problem 1.1.6

i) Prove that a number is a multiple of 4 if and only if its two last figures make up a multiple of 4.

ii) Prove that a number is a multiple of 2^k if and only if its k last figures make up a multiple of 2^k.

iii) Prove that a number is a multiple of 3 (or 9) if and only if the sum of its figures is a multiple of 3 (or 9). In other terms, $n = \sum_{j=0}^{N} a_j 10^j$ is a multiple of 3 (or 9) if and only if $\sum_{j=0}^{N} a_j$ is a multiple of 3 (or 9).

iv) Prove that a number is a multiple of 11 if and only if the sum of its even placed figures minus the sum of its odd placed figures is a multiple of 11, i.e.: $\sum_{j=0}^{N}(-1)^j a_j$ is a multiple of 11.

Hints: *ii*) write the number in the form $n = 10^k a + b$, with $a \geq 0$, $0 \leq b < 10^k$; *iii*) and *iv*) use problem 1.1.5.

Problem 1.1.7 Prove by induction and using Newton's binomial that for all $n \in \mathbb{N}$ we have that

i) $n^3 - n$ is a multiple of 6;

ii) $n^5 - n$ is a multiple of 5.

Problem 1.1.8 Prove that:

i) if $n \in \mathbb{N}$ is not a perfect square, $\sqrt{n} \notin \mathbb{Q}$;

ii) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

Hint: *i*) write $n = z^2 r$, where r does not contain any square factor.

Problem 1.1.9 Find the set of $x \in \mathbb{R}$ that verify:

i) $A = \{\, |x-3| \leq 8 \,\}$, *ii)* $B = \{\, 0 < |x-2| < 1/2 \,\}$,

iii) $C = \{\, x^2 - 5x + 6 \geq 0 \,\}$, *iv)* $D = \{\, x^3(x+3)(x-5) < 0 \,\}$,

v) $E = \{\, \dfrac{2x+8}{x^2+8x+7} > 0 \,\}$, *vi)* $F = \{\, \dfrac{4}{x} < x \,\}$,

vii) $G = \{\, 4x < 2x + 1 \leq 3x + 2 \,\}$, *viii)* $H = \{\, |x^2 - 2x| < 1 \,\}$,

ix) $I = \{\, |x-1||x+2| = 10 \,\}$, *x)* $J = \{\, |x-1| + |x-2| > 1 \,\}$.

Problem 1.1.10 Given two real numbers $a < b$, let us define, for each $t \in \mathbb{R}$ the number $x(t) = (1-t)a + tb$. Describe the following sets of numbers:

i) $A = \{\, x(t) \,:\, t = 0,\, 1,\, 1/2 \,\}$, *ii)* $B = \{\, x(t) \,:\, t \in (0,1) \,\}$,

iii) $C = \{\, x(t) \,:\, t < 0 \,\}$, *iv)* $D = \{\, x(t) \,:\, t > 1 \,\}$.

Problem 1.1.11 Find the supremum, the infimum, the maximum and the minimum (if they exist) of the following sets of real numbers:

i) $A = \{-1\} \cup [2, 3)$;

ii) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0, 1]$;

iii) $C = \{\, x = 2 + 1/n \,:\, n \in I\!N \,\}$;

iv) $D = \{\, x = (n^2 + 1)/n \,:\, n \in I\!N \,\}$;

v) $E = \{\, x \in I\!R \,:\, 3x^2 - 10x + 3 \leq 0 \,\}$;

vi) $F = \{\, x \in I\!R \,:\, (x - a)(x - b)(x - c)(x - d) < 0 \,\}$, with $a < b < c < d$ fixed;

vii) $G = \{\, x = 2^{-p} + 5^{-q} \,:\, p, q \in I\!N \,\}$;

viii) $H = \{\, x = (-1)^n + 1/m \,:\, n, m \in I\!N \,\}$.

Problem 1.1.12 Represent in $I\!R^2$ the following sets:

i) $A = \{\, |x - y| < 1 \,\}$, *ii)* $B = \{\, x^2 < y < x \,\}$,

iii) $C = \{\, x + y \in Z\!\!\!Z \,\}$, *iv)* $D = \{\, |2x| + |y| = 1 \,\}$,

v) $E = \{\, (x - 1)^2 + (y + 2)^2 < 4 \,\}$, *vi)* $F = \{\, |1 - x| = |y - 1| \,\}$,

vii) $G = \{\, 4x^2 + y^2 \leq 4,\ xy \geq 0 \,\}$, *viii)* $H = \{\, 1 \leq x^2 + y^2 < 9,\ y \geq 0 \,\}$.

Problem 1.1.13 Prove that the straight lines $y = mx + b$, $y = nx + c$ are orthogonal if $mn = -1$.

Problem 1.1.14 Consider the plane triangle defined by the points $(a, 0)$, $(-b, 0)$ and $(0, c)$, with $a, b, c > 0$.

i) Compute the intersection point of the three altitudes.

ii) Calculate the intersection point of the three medians.

iii) When do these two points coincide?

Problem 1.1.15

i) Consider the parabola $G = \{y = x^2\}$ and the point $P = (0, 1/4)$. Find $\lambda \in I\!R$ such that the points of G are equidistant from P and the horizontal line $L = \{y = \lambda\}$.

ii) Conversely, find that the set G such that its points are equidistant from a point $P = (a, b)$ and a straight line $L = \{y = \lambda\}$, is the parabola $y = \alpha x^2 + \beta x + \gamma$. Find α, β, γ.

Problem 1.1.16

i) Find the set of points in the plane such that the sum of their distances to the points $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ is $2a$, $(a > c)$.

ii) Same question, substituting sum by difference (with $a < c$).

1.2 Elementary functions

Problem 1.2.1 Find the domain of the following functions:

$$i)\quad f(x)=\frac{1}{x^2-5x+6}, \qquad ii)\quad f(x)=\sqrt{1-x^2}+\sqrt{x^2-1},$$

$$iii)\quad f(x)=\frac{1}{x-\sqrt{1-x^2}}, \qquad iv)\quad f(x)=\sqrt{1-\sqrt{4-x^2}},$$

$$v)\quad f(x)=\frac{1}{1-\log x}, \qquad vi)\quad f(x)=\log(x-x^2),$$

$$vii)\quad f(x)=\frac{\sqrt{5-x}}{\log x}, \qquad viii)\quad f(x)=\arcsin(\log x).$$

Problem 1.2.2

i) If both f and g are odd functions, what are $f+g$, $f\cdot g$ and $f\circ g$?

ii) What if f is even and g is odd?

Problem 1.2.3 Study the symmetry of the following functions:

$$i)\quad f(x)=\frac{x}{x^2-1}, \qquad ii)\quad f(x)=\frac{x^2-x}{x^2+1},$$

$$iii)\quad f(x)=\frac{\sin x}{x}, \qquad iv)\quad f(x)=(\cos x^3)(\sin x^2)\mathrm{e}^{-x^4},$$

$$v)\quad f(x)=\frac{1}{\sqrt{x^2+1}-x}, \qquad vi)\quad f(x)=\log(\sqrt{x^2+1}-x).$$

Hint: $vi)$ is odd.

Problem 1.2.4 For which numbers $a,b,c,d\in\mathbb{R}$ does the function $f(x)=\dfrac{ax+b}{cx+d}$ satisfy $f\circ f=I$ (identity) on the domain of f?

Problem 1.2.5 Check that the function $f(x)=\dfrac{x+3}{1+2x}$ is bijective, defined from $\mathbb{R}-\{-1/2\}$ to $\mathbb{R}-\{1/2\}$ and find its inverse.

Problem 1.2.6

i) Study which of the following functions are injective, finding their inverses when they have them, or give an example of two points with the same image otherwise.

$$a)\quad f(x)=7x-4, \qquad b)\quad f(x)=\sin(7x-4),$$

$$c)\quad f(x)=(x+1)^3+2, \qquad d)\quad f(x)=\frac{x+2}{x+1},$$

$$e)\quad f(x)=x^2-3x+2, \qquad f)\quad f(x)=\frac{x}{x^2+1},$$

$$g)\quad f(x)=\mathrm{e}^{-x}, \qquad h)\quad f(x)=\log(x+1).$$

ii) Prove that the function $f(x) = x^2 - 3x + 2$ is injective on $(3/2, \infty)$.

iii) Prove that the function $f(x) = \dfrac{x}{x^2+1}$ is injective on $(1, \infty)$ and find $f^{-1}(\sqrt{2}/3)$.

iv) Study whether the previous functions are surjective and bijective on their domain $D(f)$ in $I\!R$.

Problem 1.2.7 Prove that $a \sin x + b \cos x$ can be written as $A \sin(x + B)$, and find A and B.

Problem 1.2.8 Calculate

$$i) \qquad \operatorname{arctg} \frac{1}{2} + \operatorname{arctg} \frac{1}{3},$$

$$ii) \qquad \operatorname{arctg} 2 + \operatorname{arctg} 3,$$

$$iii) \qquad \operatorname{arctg} \frac{1}{2} + \operatorname{arctg} \frac{1}{5} + \operatorname{arctg} \frac{1}{8}.$$

Hint: use the formula for the tangent of a sum and study the signs.

Problem 1.2.9 Simplify the following expressions

$$i) \quad f(x) = \sin(\arccos x), \qquad ii) \quad f(x) = \sin(2 \arcsin x),$$

$$iii) \quad f(x) = \operatorname{tg}(\arccos x), \qquad iv) \quad f(x) = \sin(2 \operatorname{arctg} x),$$

$$v) \quad f(x) = \cos(2 \operatorname{arctg} x), \qquad vi) \quad f(x) = \mathrm{e}^{4 \log x}.$$

Problem 1.2.10 Solve the following system of equations, for $x, y > 0$,

$$\begin{cases} x^y = y^x \\ y = 3x. \end{cases}$$

Problem 1.2.11 Describe function g in terms of f in the following cases ($c \in I\!R$ is a constant). Plot them for $f(x) = x^2$ and $f(x) = \sin x$.

$$i) \quad g(x) = f(x) + c, \qquad ii) \quad g(x) = f(x + c),$$

$$iii) \quad g(x) = f(cx), \qquad iv) \quad g(x) = f(1/x),$$

$$v) \quad g(x) = f(|x|), \qquad vi) \quad g(x) = |f(x)|,$$

$$vii) \quad g(x) = 1/f(x), \qquad viii) \quad g(x) = (f(x))_+ = \max\{f(x), 0\}.$$

Problem 1.2.12 Sketch, with as few calculations as possible, the graph of the following functions:

$$i)\quad f(x) = (x+2)^2 - 1, \qquad ii)\quad f(x) = \sqrt{4-x},$$

$$iii)\quad f(x) = x^2 + 1/x, \qquad iv)\quad f(x) = 1/(1+x^2),$$

$$v)\quad f(x) = \min\{x, x^2\}, \qquad vi)\quad f(x) = |e^x - 1|,$$

$$vii)\quad f(x) = \sqrt{x - [x]}, \qquad viii)\quad f(x) = 1/[1/x],$$

$$ix)\quad f(x) = |x^2 - 1|, \qquad x)\quad f(x) = 1 - e^{-x},$$

$$xi)\quad f(x) = \log(x^2 - 1), \qquad xii)\quad f(x) = x\sin(1/x).$$

Hint: $[x] = n$ denotes the integer part of x, i.e., the biggest integer $n \le x$.

Problem 1.2.13 Let us define the hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

i) Study their symmetry.

ii) Prove the formulas

$$a)\ \cosh^2 x - \sinh^2 x = 1, \qquad b)\ \sinh 2x = 2 \sinh x \cosh x.$$

iii) Simplify the function $f(x) = \sinh^{-1} x$.

iv) Sketch the graph of the functions $\sinh x$ and $\cosh x$.

Problem 1.2.14 Sketch the following curves given in polar coordinates:

$$i)\quad r = 1, \quad \theta \in [0, \pi], \qquad ii)\quad \theta = 3\pi/4, \quad r \ge 2,$$

$$iii)\quad r = 2\sin\theta, \quad \theta \in [0, \pi], \qquad iv)\quad r = \theta, \quad \theta \in [0, 2\pi],$$

$$v)\quad r = e^\theta, \quad \theta \in [-2\pi, 2\pi], \qquad vi)\quad r = \sec\theta, \quad \theta \in [0, \pi/2],$$

$$vii)\quad r = 1 - \sin\theta, \quad \theta \in [0, 2\pi], \qquad viii)\quad r = (\cos 2\theta)_+, \quad \theta \in [0, 2\pi],$$

$$ix)\quad r = |\cos 2\theta|, \quad \theta \in [0, 2\pi], \qquad x)\quad r = (\sin 3\theta)_+, \quad \theta \in [0, 2\pi/3].$$

Problem 1.2.15 Sketch the following subsets of the plane given in polar coordinates:

$$i)\quad A = \{\, 1 < r < 4 \,\}, \qquad ii)\quad B = \{\, \pi/6 \le \theta \le \pi/3 \,\},$$

$$iii)\quad C = \{\, r \le \theta,\ 0 \le \theta \le 3\pi/2 \,\}, \qquad iv)\quad D = \{\, r \le \sec\theta,\ 0 \le \theta \le \pi/4 \,\}.$$

1.3 Limits of functions

Problem 1.3.1 Using the ε–δ definition of limit, prove that:

$$i)\ \lim_{x\to 2} x^2 = 4, \qquad ii)\ \lim_{x\to 3}(5x-1) \neq 16,$$

$$iii)\ \lim_{x\to 0} \frac{x}{1+\sin^2 x} = 0, \qquad iv)\ \lim_{x\to 9} \sqrt{x} = 3.$$

Problem 1.3.2 Find the following limits simplifying the common factors which might appear:

$$i)\ \lim_{x\to a} \frac{x^n - a^n}{x-a}, \qquad ii)\ \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a},$$

$$iii)\ \lim_{x\to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}, \qquad iv)\ \lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2},$$

$$v)\ \lim_{h\to 0} \frac{\frac{1}{(1-h)^3}-1}{h}, \qquad vi)\ \lim_{x\to 1} \Big(\frac{1}{\sqrt{x}-1} - \frac{2}{x-1}\Big).$$

Problem 1.3.3 Using the limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, find the following limits:

$$i)\ \lim_{x\to 0} \frac{(\sin 2x^3)^2}{x^6}, \qquad ii)\ \lim_{x\to 0} \frac{1-\cos x}{x^2},$$

$$iii)\ \lim_{x\to 0} \frac{\operatorname{tg} x^2 + 2x}{x+x^2}, \qquad iv)\ \lim_{x\to 0} \frac{\sin(x+a)-\sin a}{x},$$

$$v)\ \lim_{x\to 0} \frac{\log(1+x)}{x}, \qquad vi)\ \lim_{x\to 0} (1+x)^{1/x},$$

$$vii)\ \lim_{x\to 0} \frac{\log(1-2x)}{\sin x}, \qquad viii)\ \lim_{x\to 0} (1+\sin x)^{2/x},$$

$$ix)\ \lim_{x\to 0} \frac{e^x - e^{\sin x}}{x-\sin x}, \qquad x)\ \lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3},$$

$$xi)\ \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}}, \qquad xii)\ \lim_{x\to 0} (\cos x)^{1/x^2},$$

$$xiii)\ \lim_{x\to \pi} \frac{1-\sin(x/2)}{(x-\pi)^2}, \qquad xiv)\ \lim_{x\to 0} \frac{a^x - b^x}{x}.$$

Hint: it may be necessary to use a change of variables and the limit of the composite function.

Problem 1.3.4 Calculate the following limits:

$$i) \quad \lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}, \qquad ii) \quad \lim_{x\to\infty} \frac{x+\sin x^3}{5x+6},$$

$$iii) \quad \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}, \qquad iv) \quad \lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right),$$

$$v) \quad \lim_{x\to\infty} \frac{e^x}{e^x-1}, \qquad vi) \quad \lim_{x\to-\infty} \frac{e^x}{e^x-1},$$

$$vii) \quad \lim_{x\to\infty} \frac{x-2}{\sqrt{4x^2+1}}, \qquad viii) \quad \lim_{x\to-\infty} \frac{x-2}{\sqrt{4x^2+1}}.$$

Problem 1.3.5 Find the one-sided limits:

$$i) \quad \lim_{t\to 0^+} \left(\frac{1}{t}\right)^{[t]}, \qquad ii) \quad \lim_{t\to 0^-} \left(\frac{1}{t}\right)^{[t]},$$

$$iii) \quad \lim_{t\to 0^+} e^{1/t}, \qquad iv) \quad \lim_{t\to 0^-} e^{1/t}.$$

Problem 1.3.6 Find the limits

$$i) \lim_{x\to-\infty} \left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}, \qquad ii) \lim_{t\to 0} \frac{1-e^{1/t}}{1+e^{1/t}}$$

Problem 1.3.7

i) Establish the relation between a and b so that

$$\lim_{x\to 1} x^{a/(1-x)} = \lim_{x\to 0} (\cos x)^{b/x^2}.$$

ii) If $f(x) = \log(\log x)$ and $\alpha > 0$, find $\lim_{x\to\infty} (f(x) - f(\alpha x))$ and $\lim_{x\to\infty} (f(x) - f(x^\alpha))$.

Problem 1.3.8

i) Prove that if $\lim_{x\to 0} f(x) = 0$ then $\lim_{x\to 0} f(x)\sin 1/x = 0$.

ii) Calculate $\lim_{x\to 0} \dfrac{x}{2+\sin 1/x}$.

1.4 Continuity

Problem 1.4.1

i) Prove that if f is continuous at a point a and g is at $f(a)$, then $g \circ f$ is continuous at a.

ii) Prove that if f is continuous, then $|f|$ is also. Is the reciprocal true?

iii) What can be said of a function that only takes values on $\mathbb{Q}$?

Problem 1.4.2 Find $\lambda \in \mathbb{R}$ so that the function $b(x) = \dfrac{1}{\lambda x^2 - 2\lambda x + 1}$ is continuous on:

$$i) \ \mathbb{R}, \qquad ii) \ [0,1].$$

Problem 1.4.3 Study the continuity of the following functions:

i) $f(x) = \dfrac{e^{-5x} + \cos x}{x^2 - 8x + 12}$;

ii) $g(x) = e^{3/x} + x^3 - 9$;

iii) $h(x) = x^3 \operatorname{tg}(3x + 2)$;

iv) $j(x) = \sqrt{x^2 - 5x + 6}$;

v) $k(x) = (\arcsin x)^3$;

vi) $m(x) = (x - 5)\log(8x - 3)$;

Problem 1.4.4 Study the continuity of the following functions:

i) $f(x) = x - [x]$;

ii) $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0; \end{cases}$

iii) $f(x) = \begin{cases} \dfrac{\operatorname{tg} x}{\sqrt{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ e^{1/x} & \text{if } x < 0; \end{cases}$

iv) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q}. \end{cases}$

Problem 1.4.5 Prove the following fixed points theorems:

i) Let $f : [0,1] \longrightarrow [0,1]$ a continuous function. Then there exists $c \in [0,1]$ such that $f(c) = c$.

ii) Let $f,\, g : [a,b] \longrightarrow \mathbb{R}$ two continuous functions such that $f(a) > g(a)$, $f(b) < g(b)$. Then there exists $c \in (a,b)$ such that $f(c) = g(c)$.