3 Transformada de Laplace.

3.1 Cálculo de transformadas y antitransformadas

Problema 3.1.1

i) $F(s) = \dfrac{d^2}{ds^2}\left(\dfrac{s+1}{(s+1)^2+1}\right) = \dfrac{2(s+1)(s^2+2s-2)}{(s^2+2s+2)^3}.$

ii) $F(s) = \dfrac{3}{4}\dfrac{s}{s^2+1} + \dfrac{1}{4}\dfrac{s}{s^2+9} = \dfrac{s(s^2+7)}{(s^2+1)(s^2+9)}.$

iii) $F(s) = \displaystyle\int_s^\infty \frac{1}{\sigma^2+1}\,d\sigma = \frac{\pi}{2} - \operatorname{arctg} s.$

iv) $F(s) = -\dfrac{d}{ds}\left(\dfrac{1}{s[(s+1)^2+1]}\right) = \dfrac{3s^2+4s+2}{s^2(s^2+2s+2)^2}.$

Problema 3.1.2

i) $f(t) = t - \displaystyle\sum_{n=1}^\infty H(t-n) \;\rightsquigarrow\; F(s) = \frac{1}{s^2} - \sum_{n=1}^\infty \frac{\mathrm{e}^{-ns}}{s} = \frac{1-(s+1)\mathrm{e}^{-s}}{s^2(1-\mathrm{e}^{-s})}.$

O también, $f(t) = t$ periódica con $T = 1 \;\rightsquigarrow\; F(s) = \dfrac{1}{1-\mathrm{e}^{-s}}\displaystyle\int_0^1 t\mathrm{e}^{-st}\,dt.$

ii) $f(t) = (t-1)^k H(t-1) \;\rightsquigarrow\; F(s) = \dfrac{k!}{s^{k+1}}\,\mathrm{e}^{-s}.$

iii) $f(t) = (t-1+1)^k H(t-1) = \displaystyle\sum_{n=0}^k \binom{k}{n}(t-1)^n H(t-1) \;\rightsquigarrow\; F(s) = \sum_{n=0}^k \frac{k!}{(k-n)!\,s^{n+1}}\,\mathrm{e}^{-s}.$

iv) $f(t) = 1 + 2\displaystyle\sum_{n=1}^\infty (-1)^n H(t-n) \;\rightsquigarrow\; F(s) = \frac{1-\mathrm{e}^{-s}}{s(1+\mathrm{e}^{-s})}.$

O también $f(t) = 1$ si $0 \le t < 1$, $f(t) = -1$ si $1 \le t < 2$, periódica con $T = 2$.

Problema 3.1.3

i) $F(s) = \dfrac{1}{4}\Big(\dfrac{1}{s} - \dfrac{s}{s^2+4}\Big)\mathrm{e}^{-3s} \;\rightsquigarrow\; f(t) = \dfrac{1}{4}(1-\cos 2(t-3))H(t-3).$

ii) $f(t) = \mathrm{e}^{-t}(\cos 2t + \operatorname{sen} 2t)$. *iii)* $f(t) = \dfrac{1}{9}(1 - \cos 3t + 6\operatorname{sen} 3t).$

iv) $f(t) = 2\mathrm{e}^{-2t}\operatorname{sen} 3t$. *v)* $f(t) = t - \operatorname{senh} t$. *vi)* $\left(\dfrac{s}{s^2+k^2}\right)' = \dfrac{2k^2}{(s^2+k^2)^2} - \dfrac{1}{s^2+k^2}.$

$$f(t) = \frac{1}{2k^2}\left(-tL^{-1}\Big(\frac{s}{s^2+k^2}\Big) + L^{-1}\Big(\frac{1}{s^2+k^2}\Big)\right) = \frac{1}{2k^3}(\operatorname{sen} kt - kt\cos kt).$$

Problema 3.1.4 $g(t) = f(t)_+ = \left\{\begin{array}{ll} f(t) & \text{si } f(t) \ge 0 \\ 0 & \text{si } f(t) \le 0 \end{array}\right\} = \left\{\begin{array}{ll} \operatorname{sen} kt & \text{si } 0 \le t \le \pi/k \\ 0 & \text{si } \pi/k \le t \le 2\pi/k \end{array}\right\},$

periódica de periodo $T = \dfrac{2\pi}{k}$; $G(s) = \dfrac{1}{1-\mathrm{e}^{-2\pi s/k}}\displaystyle\int_0^{\pi/k} \operatorname{sen} kt\,\mathrm{e}^{-st}\,dt = \frac{k}{(1-\mathrm{e}^{-\pi s/k})(s^2+k^2)}.$

Problema 3.1.5

i) $\displaystyle\int_0^\infty F(s)\,ds = \int_0^\infty \int_0^\infty \mathrm{e}^{-sx} f(x)\,dx\,ds = \int_0^\infty f(x) \int_0^\infty \mathrm{e}^{-sx}\,ds\,dx = \int_0^\infty \frac{f(x)}{x}\,dx.$

ii) 1) $\displaystyle\int_0^\infty \frac{\operatorname{sen} x}{x}\,dx = \int_0^\infty \frac{1}{s^2+1}\,ds = \frac{\pi}{2}.$

2) $\displaystyle\int_0^\infty \frac{\mathrm{e}^{-\alpha x} - \mathrm{e}^{-\beta x}}{x}\,dx = \int_0^\infty (\frac{1}{s+\alpha} - \frac{1}{s+\beta})\,ds = \log(\beta/\alpha).$

Problema 3.1.6 $\displaystyle f(t) = \sum_{n=0}^\infty a_n t^n \Rightarrow F(s) = \sum_{n=0}^\infty a_n \frac{n!}{s^{n+1}}$

i) $\displaystyle F(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \frac{(2n)!}{s^{2n+1}} = \frac{s}{s^2+1}.$

ii) $\displaystyle F(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \frac{(2n)!}{s^{2n+1}} = \operatorname{arctg}(1/s) = \frac{\pi}{2} - \operatorname{arctg} s.$

iii) $\displaystyle F(s) = \sum_{n=0}^\infty \frac{1}{(n+1)s^{n+1}} = \log(\frac{s}{s-1}).$

iv) $\displaystyle F(s) = \sum_{n=0}^\infty \frac{(-1)^n (2n)!\, s^{-2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^\infty \binom{-1/2}{n} s^{-2n-1} = \frac{1}{\sqrt{s^2+1}}.$

v) $\displaystyle F(s) = \sum_{n=0}^\infty \frac{(-1)^n s^{-n-1}}{2^{2n} n!} = \frac{\mathrm{e}^{-1/4s}}{s}.$

Problema 3.1.7

i) $\displaystyle f(t) = \sum_{n=0}^\infty \frac{(-1)^n t^n}{(n+1)!} = \frac{1-\mathrm{e}^{-t}}{t}.$

ii) $\displaystyle f(t) = \sum_{n=0}^\infty \frac{2(-1)^n t^{2n+1}}{(2n+2)!} = \frac{2(1-\cos t)}{t}.$

Problema 3.1.8 Sea $x > 0$. $\displaystyle Y(s) = \int_0^\infty \int_0^\infty \frac{\cos xt\, \mathrm{e}^{-sx}}{1+t^2}\,dt\,dx = \int_0^\infty \frac{1}{1+t^2} \frac{s}{s^2+t^2}\,dt = \frac{s}{s^2-1} \int_0^\infty \left(\frac{1}{t^2+1} - \frac{1}{t^2+s^2}\right) dt = \frac{\pi}{2(s+1)}$; la antitransformada es entonces $y(x) = \frac{\pi}{2}\mathrm{e}^{-x}$. Por simetría la integral sería igual sustituyendo x por $|x|$.

3.2 Resolución de ecuaciones mediante transformada de Laplace

Problema 3.2.1

i) $\displaystyle Y(s) = \frac{1}{s-5}\left(\frac{1}{2} + \frac{s}{s^2+9}\right)$; $\displaystyle y(t) = \frac{1}{34}(22\mathrm{e}^{5t} - 5\cos 3t + 3\operatorname{sen} 3t).$

ii) $\displaystyle Y(s) = \frac{1}{s^2-1}\left(1 + \frac{1}{s-2}\right)$; $\displaystyle y(t) = \frac{1}{3}(\mathrm{e}^{2t} - \mathrm{e}^{-t}).$

iii) $Y(s) = \dfrac{1}{s^2+16}\left(1 + \dfrac{s}{s^2+16}\right)$; $y(t) = \dfrac{1}{8}(2\operatorname{sen} 4t + t \operatorname{sen} 4t)$.

iv) $Y(s) = \dfrac{1}{s^2+2s+1}\left(s+1+\dfrac{1}{s+3}\right)$; $y(t) = \dfrac{1}{4}[(2t+3)\mathrm{e}^{-t} + \mathrm{e}^{-3t}]$.

v) $Y(s) = \dfrac{1}{s^2-6s+9}\left(2s-6+\dfrac{2}{(s-3)^2}\right)$; $y(t) = \dfrac{1}{12}(t^4+24)\mathrm{e}^{3t}$.

vi) $Y(s) = \dfrac{1}{s^2+4s+6}\left(\dfrac{1}{s}+\dfrac{1}{s+1}\right)$; $y(t) = \dfrac{1}{6}[1+2\mathrm{e}^{-t} - (3\cos\sqrt{2}t + 2\sqrt{2}\operatorname{sen}\sqrt{2}t)\mathrm{e}^{-2t}]$.

Problema 3.2.2

$$\begin{cases} (s-6)X + 3Y = \dfrac{8}{s-1} - 1 \\ (s-1)Y - 2X = \dfrac{4}{s-1} \end{cases} \quad \rightsquigarrow \quad \begin{cases} x(t) = \mathrm{e}^{4t} - 2\mathrm{e}^{t} - \dfrac{1}{2}\mathrm{e}^{3t} \\ y(t) = (\dfrac{4}{2} - \dfrac{7}{3}t)\mathrm{e}^{t} + 8\mathrm{e}^{4t} - \dfrac{21}{2}\mathrm{e}^{3t} \end{cases}$$

Problema 3.2.3

i) El término independiente es $f(t) = t + 1 - 2(t-1)H(t-1)$;
la ecuación transformada es $(s^2+s)Y = -1 - s + \dfrac{1}{s} + \dfrac{1}{s^2} - \dfrac{2}{s^2}\mathrm{e}^{-s}$;
la solución es $y(t) = \dfrac{t^2}{2} - 1 - (t^2 - 4t + 5 - 2\mathrm{e}^{1-t})H(t-1) = \begin{cases} \dfrac{t^2}{2} - 1 & \text{si} \quad 0 \le t \le 1 \\ -\dfrac{t^2}{2} + 4t - 6 + 2\mathrm{e}^{1-t} & \text{si} \quad t > 1 \end{cases}$

ii) El término independiente es $f(t) = \cos 2t(1 - H(t-2\pi))$;
la ecuación transformada es $(s^2+4)Y = \dfrac{s}{s^2+4}(1 - \mathrm{e}^{-2\pi s})$;
la solución es $y(t) = \dfrac{1}{4}\operatorname{sen} 2t\Big(t - (t-2\pi)H(t-2\pi)\Big)$.

Problema 3.2.4 El término independiente es $f(t) = \mathrm{e}^{\pi/2}\mathrm{e}^{t-\pi/2}H(t-\pi/2) - \mathrm{e}^{\pi}\mathrm{e}^{t-\pi}H(t-\pi)$;
la ecuación transformada es $(s^2-1)X = \dfrac{1}{s-1}(\mathrm{e}^{\pi(1-s)/2} - \mathrm{e}^{\pi(1-s)})$;
la solución es $x(t) = \dfrac{1}{4}(\mathrm{e}^{\pi-t} - \mathrm{e}^{t} + 2(t-\pi/2)\mathrm{e}^{t})H(t-\pi/2) - \dfrac{1}{4}(\mathrm{e}^{2\pi-t} - \mathrm{e}^{t} + 2(t-\pi)\mathrm{e}^{t})H(t-\pi)$.

Problema 3.2.5 $G_a(t) = H(t) - H(t-a)$; $L(G_a)(s) = \dfrac{1-\mathrm{e}^{-as}}{s}$.

i)
$$\begin{aligned} y(t) &= \frac{1}{2}[(\operatorname{senh} t - \operatorname{sen} t) + (\cosh(t-1) + \cos(t-1) - 2)H(t-1) \\ &\quad -(\cosh(t-4) + \cos(t-4) - 2)H(t-4)]. \end{aligned}$$

ii) $y(t) = \mathrm{e}^{-t} + \operatorname{senh}(t-3)H(t-3) - \operatorname{senh}(t-7)H(t-7)$.

Problema 3.2.6 $y_n(t) = \Big(\dfrac{1}{n+1}(n\mathrm{e}^{t-1} + \mathrm{e}^{-n(t-1)}) - 1\Big)H(t-1)$;
$\lim\limits_{n\to\infty} y_n(t) = ((\mathrm{e}^{t-1} - 1)H(t-1)$, que es la solución del problema con fuente $\lim\limits_{n\to\infty} f_n(t) = \delta(t-1)$.

Problema 3.2.7

i)

$$1)\ \begin{cases} (s+1)X+Y=1 \\ (s+1)Y-2X=2 \end{cases} \rightsquigarrow \begin{cases} x(t)=\mathrm{e}^{-t}(\cos\sqrt{2}t-\sqrt{2}\,\mathrm{sen}\,\sqrt{2}t) \\ y(t)=\mathrm{e}^{-t}(2\cos\sqrt{2}t+\sqrt{2}\,\mathrm{sen}\,\sqrt{2}t) \end{cases}$$

$$2)\ \begin{cases} (s-1)X+2Y=\dfrac{2}{s} \\ (s+1)Y-5X=\dfrac{1}{s} \end{cases} \rightsquigarrow \begin{cases} x(t)=\dfrac{2}{3}\,\mathrm{sen}\,3t \\ y(t)=\dfrac{1}{3}\,\mathrm{sen}\,3t-\cos 3t+1 \end{cases}$$

$$3)\ \begin{cases} sX-x_0=X \\ sY-y_0=X+2Y \\ sZ-z_0=X-Z \end{cases} \rightsquigarrow \begin{cases} x(t)=x_0\mathrm{e}^{t} \\ y(t)=(x_0+y_0)\mathrm{e}^{2t}-x_0\mathrm{e}^{t} \\ z(t)=\dfrac{1}{2}(x_0\mathrm{e}^{t}+(2z_0-x_0)\mathrm{e}^{-t}) \end{cases}$$

ii)

1) $I(t)=1-\cos 2t-2(1+\cos 2t)H(t-3\pi/2)+(1-\cos 2t)H(t-3\pi)$.

2) $I(t)=(1-\cos 2t)[1-2H(t-\pi)+H(t-2\pi)]$.

Problema 3.2.8

$$\begin{cases} sX+Y=\dfrac{s+1}{s^2}\mathrm{e}^{-s}-\mathrm{e} \\ (s+2)Y-X=0 \end{cases} \rightsquigarrow \begin{cases} x(t)=-(1+t)\mathrm{e}^{1-t}+(2t-3+\mathrm{e}^{1-t})H(t-1) \\ y(t)=-t\mathrm{e}^{1-t}+(t-2+\mathrm{e}^{1-t})H(t-1) \end{cases}$$

Problema 3.2.9

$$y(t)=\begin{cases} \dfrac{1}{\sqrt{c^2-1}}\mathrm{e}^{-c(t-1)}\,\mathrm{senh}(\sqrt{c^2-1}(t-1))H(t-1) & \text{si} \quad c>1 \\[2ex] \dfrac{1}{\sqrt{1-c^2}}\mathrm{e}^{-c(t-1)}\,\mathrm{sen}(\sqrt{1-c^2}(t-1))H(t-1) & \text{si} \quad 0\le c<1 \\[2ex] (t-1)\mathrm{e}^{-(t-1)}H(t-1) & \text{si} \quad c=1 \end{cases}$$

Problema 3.2.10

i) $F(s)=s\mathrm{e}^{-as}$.

ii) $u(t)=\cos(\omega(t-a))H(t-a)$, que es discontinua en $t=a$.

iii) $u(t)=\delta(t-a)-\omega\,\mathrm{sen}(\omega(t-a))H(t-a)$, que no es una función.

Problema 3.2.11 $Y(s)=\dfrac{F(s)}{s^2+2s+2}$, de donde $y(t)=f*g(t)=\displaystyle\int_0^t f(\sigma)g(t-\sigma)\,d\sigma$, siendo $g(t)=L^{-1}(\dfrac{1}{s^2+2s+2})=\mathrm{e}^{-t}\,\mathrm{sen}\,t$. Así se tiene la representación con $G(t,\sigma)=g(t-\sigma)$.

Problema 3.2.12

i) $(s^2+1)X'(s)+1=0$.

ii) $X(s)=\dfrac{\pi}{2}-\mathrm{arctg}\,s$.

iii) $x(t)=\dfrac{\mathrm{sen}\,t}{t}$.

Problema 3.2.13

i) $L(y) + L(k)L(y) = L(f) \;\Rightarrow\; L(y) = \dfrac{L(f)}{1 + L(k)}$.

ii) $y(t) = L^{-1}\left(\dfrac{2s^2}{(s^2+1)(s^2+4)}\right) = \dfrac{2}{3}(2\operatorname{sen} 2t - \operatorname{sen} t)$. Por otro lado, derivando se tiene:

$$\begin{aligned}
&y(t) + \int_0^t (t-s)y(s)\,ds = \operatorname{sen} 2t\\
&y'(t) + \int_0^t y(s) = 2\cos 2t\\
&y''(t) + y(t) = -4\operatorname{sen} 2t
\end{aligned}$$

Los datos se obtienen de sustituir $t = 0$ en las dos primeras expresiones.

iii) Por Laplace se tiene $L(y) = \dfrac{1}{s^2} - \dfrac{1}{s} + 2\dfrac{L(y)}{s^2+1}$, que despejando y antitransformando da $y(x) = 1 - x - 2\mathrm{e}^{-x}$. Por otro lado, derivando se tiene

$$\begin{aligned}
&y(x) = x - 1 + 2\int_0^x \operatorname{sen}(x-t)y(t)\,dt\\
&y'(x) = 1 + 2\int_0^x \cos(x-t)y(t)\,dt\\
&y''(x) = 2y(x) - 2\int_0^x \operatorname{sen}(x-t)y(t)\,dt
\end{aligned}$$

Así, y sustituyendo $x = 0$ para obtener los datos iniciales, el problema equivalente es:

$$\begin{cases} y'' - y = x - 1\\ y(0) = -1,\; y'(0) = 1 \end{cases}$$

Problema 3.2.14

i) $L(f) = \dfrac{s^2+1}{(s-2)(s-1)^2}$; $\;f(x) = 5\mathrm{e}^{2x} - 4\mathrm{e}^{x} - 2x\mathrm{e}^{x}$.

ii) $\begin{cases} f'' - 2f' + f = 5\mathrm{e}^{2x}\\ f(0) = 1,\; f'(0) = 4 \end{cases}$

iii) $L(f) = \dfrac{s^2 - s + 1}{(s^2+1)^2}$; $\;f(x) = (1 - \dfrac{x}{2})\operatorname{sen} x$; $\quad \begin{cases} f'' + f = -\cos x\\ f(0) = 0,\; f'(0) = 1 \end{cases}$

Problema 3.2.15 $\begin{cases} mx'' = -kx + c\delta(t - 3\pi)\\ x(0) = -2,\; x'(0) = 0 \end{cases}$

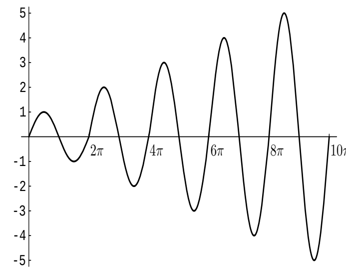
$x(t) = -2\cos(\sqrt{k/m}\,t) + \dfrac{c}{\sqrt{km}}\operatorname{sen}(\sqrt{k/m}(t - 3\pi))\,H(t - 3\pi)$.

Problema 3.2.16 $y(x) = \dfrac{A}{2}x^2 + \dfrac{B}{6}x^3 + \dfrac{L}{6EI}(x-a)^3 H(x-a) \;\Rightarrow$
$y(x) = \dfrac{L}{6EI}[3ax^2 - x^3 + (x-a)^3 H(x-a)]$.

Problema 3.2.17

i) $Y(s) = \dfrac{1}{s^2+1} \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-2k\pi s}$, que implica $y(t) = \displaystyle\sum_{k=0}^{\infty} \operatorname{sen}(t - 2k\pi) H(t - 2k\pi) = \operatorname{sen} t \sum_{k=0}^{[t/2\pi]} 1$ $= \left(\left[\dfrac{t}{2\pi} \right] + 1 \right) \operatorname{sen} t$, es decir, $y(t) = \{(k+1) \operatorname{sen} t \;\text{ si } 2k\pi \le t < 2(k+1)\pi,\ k \ge 0\}$.

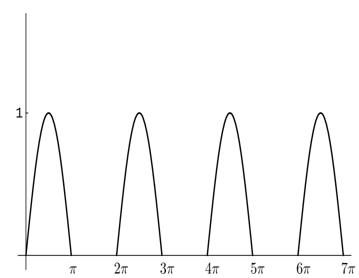
O también, $y(t) = \displaystyle\sum_{k=0}^{\infty} \int_0^t \operatorname{sen} \tau \cdot \delta(\tau - (t - 2k\pi))\, d\tau =$ $\displaystyle\sum_{k=0}^{[t/2\pi]} \operatorname{sen}(t - 2k\pi) = \operatorname{sen} t \sum_{k=0}^{[t/2\pi]} 1.$



En cada vuelta, el impulso en el sentido del movimiento hace que aumente la amplitud de oscilación.

ii) $Y(s) = \dfrac{1}{s^2+1} \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-k\pi s}$, que implica $y(t) = \displaystyle\sum_{k=0}^{\infty} \operatorname{sen}(t - k\pi) H(t - k\pi) = \operatorname{sen} t \sum_{k=0}^{[t/\pi]} (-1)^k$, es decir, $y(t) = \left\{ \begin{array}{ll} \operatorname{sen} t & \text{si } 2k\pi \le t \le (2k+1)\pi \\ 0 & \text{en el resto} \end{array} ,\ k \ge 0 \right\} = (\operatorname{sen} t)_+$. Notemos que $Y(s) = \dfrac{1}{s^2+1} \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-k\pi s} = \dfrac{1}{(s^2+1)(1 - \mathrm{e}^{-\pi s})}$, que coincide con el problema 3.1.4.

O también, $y(t) = \displaystyle\sum_{k=0}^{\infty} \int_0^t \operatorname{sen} \tau \cdot \delta(\tau - (t - k\pi))\, d\tau =$ $\displaystyle\sum_{k=0}^{[t/\pi]} \operatorname{sen}(t - k\pi) = \operatorname{sen} t \sum_{k=0}^{[t/\pi]} (-1)^k.$



En cada vuelta, un primer impulso en sentido contrario al movimiento frena éste, y un segundo impulso inicia de nuevo el movimiento.