INTRODUCCIÓN AL METODO DE LOS ELEMENTOS FINITOS

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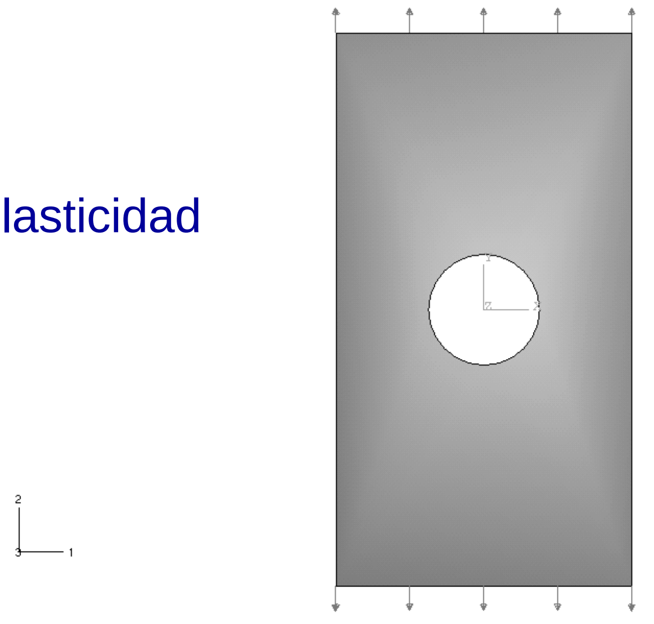
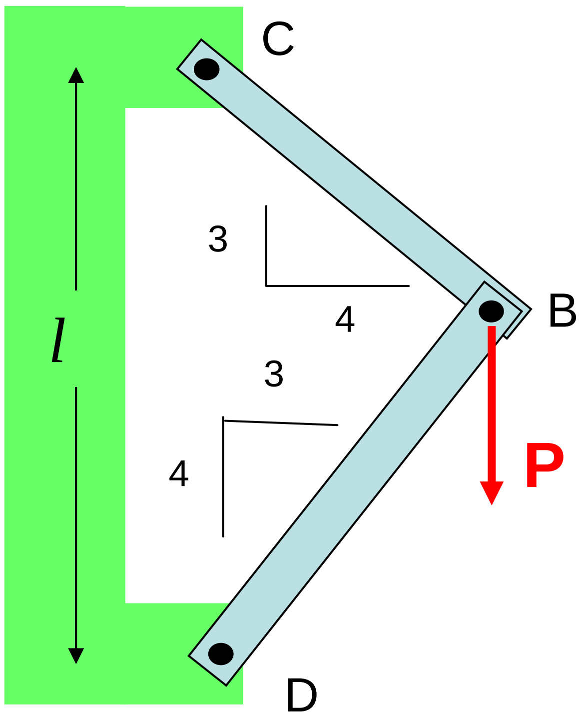
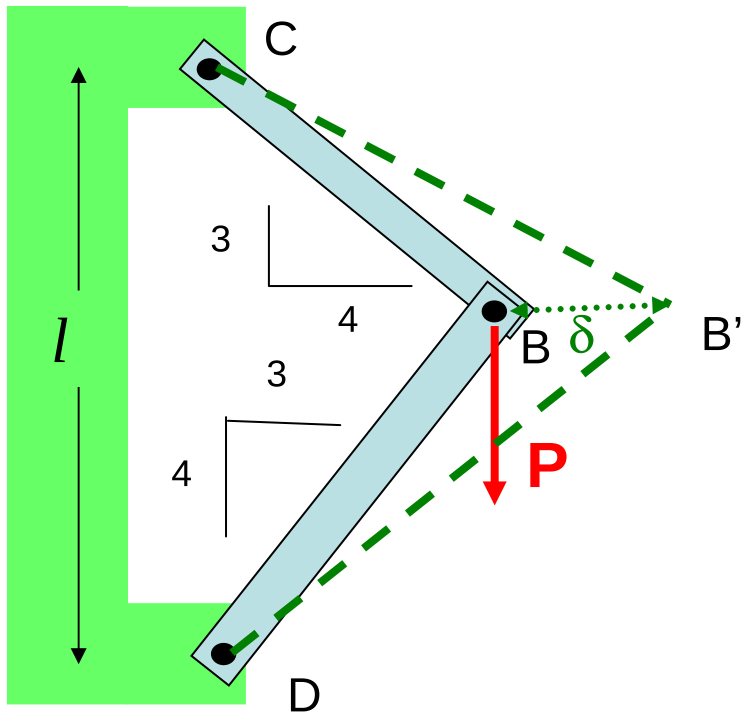
Problema de elasticidad



Modelo de elementos finitos

EJEMPLO DE APLICACIÓN DEL TEOREMA T. V.

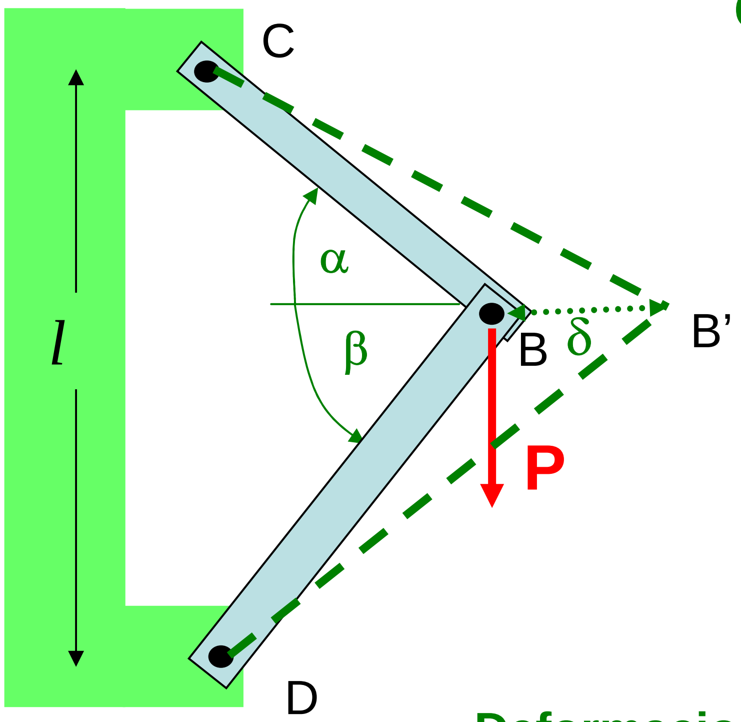


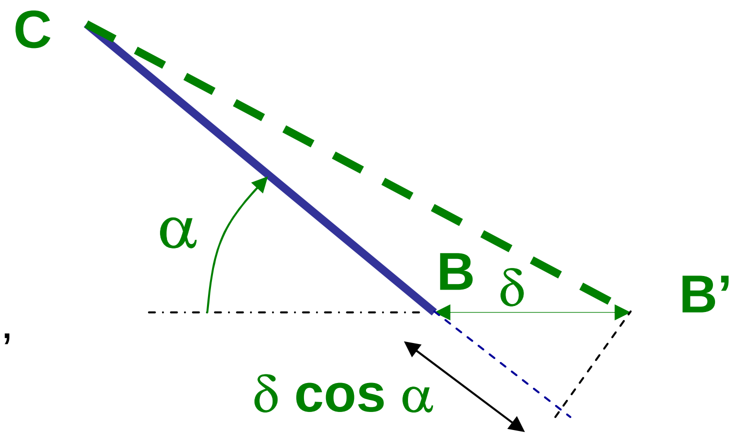


De la geometría de la estructura:

$$L_{BC} = 0{,}6\,l \quad L_{BD} = 0{,}8\,l$$

Desplazamiento virtual: **B** ⟶ **B´**





Deformaciones virtuales:

$$\begin{cases} \varepsilon_{CB}^{\delta} = \dfrac{\delta \cos \alpha}{L_{CB}} \\[2ex] \varepsilon_{DB}^{\delta} = \dfrac{\delta \cos \beta}{L_{DB}} \end{cases}$$

Trabajo virtual fuerzas exteriores:

$$W_e' = \sum \vec{F} \cdot \vec{\delta} = 0$$

Trabajo virtual desarrollado por las tensiones internas:

$$W_i' = \sum_{barras} \sigma_x \varepsilon_x^{\delta} \cdot Vol = \sigma_{CB} \frac{\delta \cos\alpha}{L_{CB}} \left(A \cdot L_{CB}\right) + \sigma_{DB} \frac{\delta \cos\beta}{L_{DB}} \left(A \cdot L_{DB}\right)$$

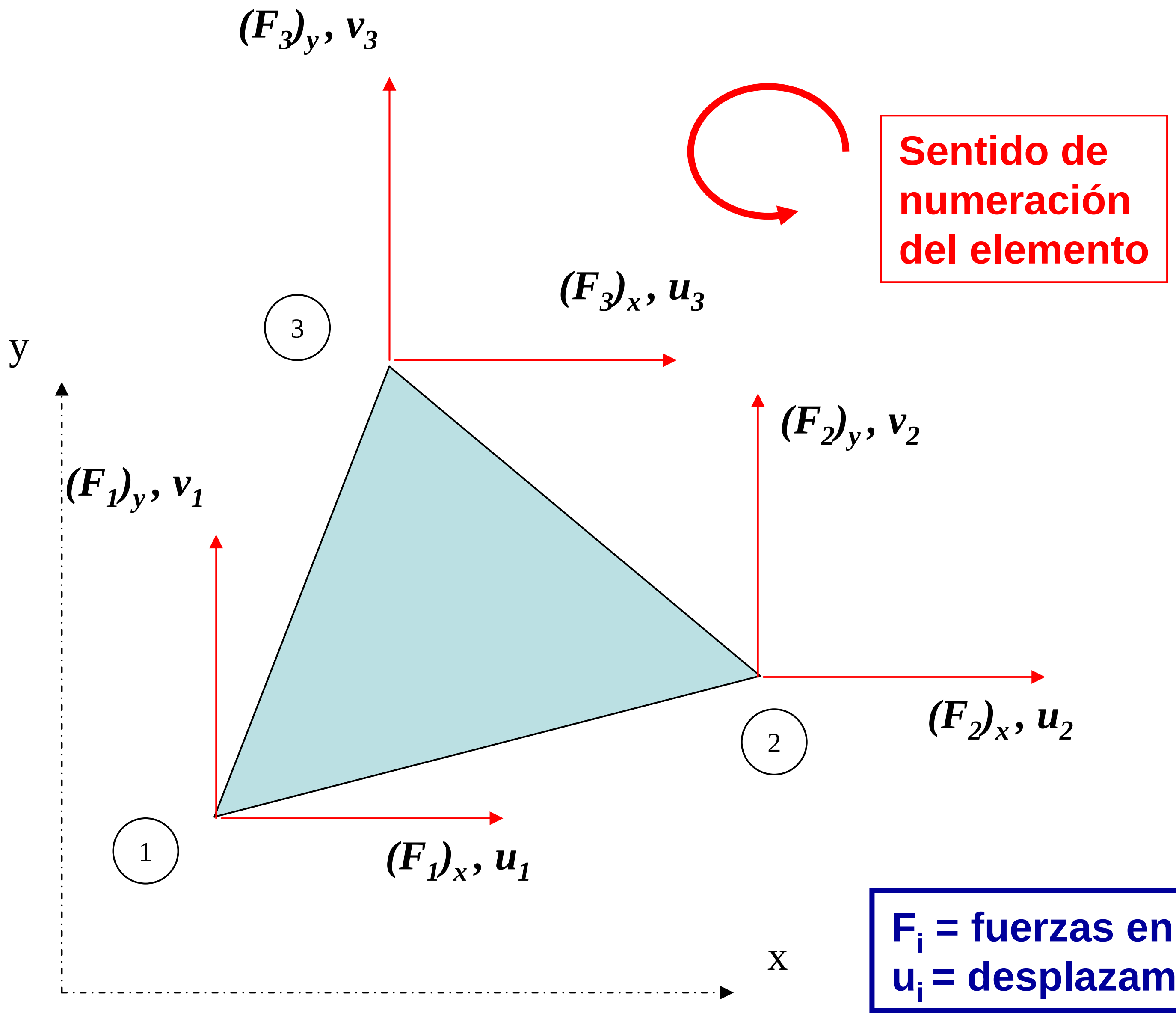
$$\sigma_{CB} \frac{\delta \cos\alpha}{L_{CB}} \left(A \cdot L_{CB}\right) + \sigma_{DB} \frac{\delta \cos\beta}{L_{DB}} \left(A \cdot L_{DB}\right) = 0$$

$$\sigma_{DB} = -\sigma_{CB} \frac{4/5}{3/5} = -\frac{4}{3} \sigma_{CB}$$

ELEMENTO TRIANGULAR DE TENSIÓN CONSTANTE

ELEMENTO DE TRES NUDOS DE TURNER (1957)

Se consideran en cada nudo dos grados de libertad de traslación



F_i = fuerzas en los nudos
u_i = desplazamientos de los nudos

¿Qué buscamos para este elemento?

$$\{F\} = \left[K^e\right]\{u\}$$

donde:

$$\{F\} = \begin{Bmatrix} (F_1)_x \\ (F_1)_y \\ (F_2)_x \\ (F_2)_y \\ (F_3)_x \\ (F_3)_y \end{Bmatrix}$$

Vector de fuerzas que actúan en los nudos

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Vector de desplazamientos de los nudos

$\left[K^e\right]$ = Matriz de rigidez del elemento (6x6)

***Hipótesis: los desplazamientos de los puntos del interior del elemento pueden aproximarse mediante funciones lineales de las coordenadas del punto que consideremos*.**

$$u(x, y) = u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v(x, y) = v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

u y ***v*** = componentes del desplazamiento en un punto genérico del elemento según los ejes coordenados

Coeficientes α**'s** = coeficientes a determinar

Lógicamente, en los nudos del elemento, dichos desplazamientos deben coincidir con los desplazamientos nodales que experimentan dichos nudos.
Si las coordenadas de los nudos fueran:
del nudo 1, *(x_1,y_1)*, del 2, *(x_2,y_2)* y del 3, *(x_3,y_3)*, se tiene que:

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$v_1 = \alpha_4 + \alpha_5 x_1 + \alpha_6 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$v_2 = \alpha_4 + \alpha_5 x_2 + \alpha_6 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$v_3 = \alpha_4 + \alpha_5 x_3 + \alpha_6 y_3$$

$$\begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \end{Bmatrix} = \begin{bmatrix} 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \boldsymbol{x}_1 & \boldsymbol{y}_1 \\ 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \boldsymbol{x}_2 & \boldsymbol{y}_2 \\ 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \boldsymbol{x}_3 & \boldsymbol{y}_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad [C] = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \quad y \quad \{\alpha\} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$$\boxed{\{u\} = [C]\{\alpha\}}$$

$$\{\alpha\} = [C]^{-1}\{u\}$$

$$[C]^{-1} = \frac{1}{2a}\begin{bmatrix} a_1 & 0 & a_2 & 0 & a_3 & 0 \\ b_1 & 0 & b_2 & 0 & b_3 & 0 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix}$$

$$a_1 = x_2 y_3 - x_3 y_2$$
$$a_2 = x_3 y_1 - x_1 y_3$$
$$a_3 = x_1 y_2 - x_2 y_1$$
$$b_1 = y_2 - y_3$$
$$b_2 = y_3 - y_1$$
$$b_3 = y_1 - y_2$$
$$c_1 = x_3 - x_2$$
$$c_2 = x_1 - x_3$$
$$c_3 = x_2 - x_1$$

$$2a = a_1 + a_2 + a_3$$

ELEMENTO DE TURNER: FUNCIONES DE FORMA

La expresión de los movimientos de un punto resulta ahora

$$\boldsymbol{u} = \alpha_1 + \alpha_2 \boldsymbol{x} + \alpha_3 \boldsymbol{y}$$

$$= \underbrace{\frac{1}{2\boldsymbol{a}}\left[\boldsymbol{a}_1 + \boldsymbol{b}_1\boldsymbol{x} + \boldsymbol{c}_1\boldsymbol{y}\right]}_{\mathbf{N_1(x,y)}} \cdot \boldsymbol{u}_1 + \underbrace{\frac{1}{2\boldsymbol{a}}\left[\boldsymbol{a}_2 + \boldsymbol{b}_2\boldsymbol{x} + \boldsymbol{c}_2\boldsymbol{y}\right]}_{\mathbf{N_2(x,y)}} \cdot \boldsymbol{u}_2 + \underbrace{\frac{1}{2\boldsymbol{a}}\left[\boldsymbol{a}_3 + \boldsymbol{b}_3\boldsymbol{x} + \boldsymbol{c}_3\boldsymbol{y}\right]}_{\mathbf{N_3(x,y)}} \cdot \boldsymbol{u}_3$$

$$\boldsymbol{v} = \alpha_4 + \alpha_5 \boldsymbol{x} + \alpha_6 \boldsymbol{y}$$

$$= \underbrace{\frac{1}{2\boldsymbol{a}}\left[\boldsymbol{a}_1 + \boldsymbol{b}_1\boldsymbol{x} + \boldsymbol{c}_1\boldsymbol{y}\right]}_{\mathbf{N_1(x,y)}} \cdot \boldsymbol{v}_1 + \underbrace{\frac{1}{2\boldsymbol{a}}\left[\boldsymbol{a}_2 + \boldsymbol{b}_2\boldsymbol{x} + \boldsymbol{c}_2\boldsymbol{y}\right]}_{\mathbf{N_2(x,y)}} \cdot \boldsymbol{v}_2 + \underbrace{\frac{1}{2\boldsymbol{a}}\left[\boldsymbol{a}_3 + \boldsymbol{b}_3\boldsymbol{x} + \boldsymbol{c}_3\boldsymbol{y}\right]}_{\mathbf{N_3(x,y)}} \cdot \boldsymbol{v}_3$$

"Los desplazamientos de un punto cualquiera son una media ponderada de los movimientos de los extremos"

Es decir:

$$u = N_1(x,y)\cdot u_1 + N_2(x,y)\cdot u_2 + N_3(x,y)\cdot u_3$$

$$v = N_1(x,y)\cdot v_1 + N_2(x,y)\cdot v_2 + N_3(x,y)\cdot v_3$$

N_i (x,y) son las denominadas funciones de forma del elemento

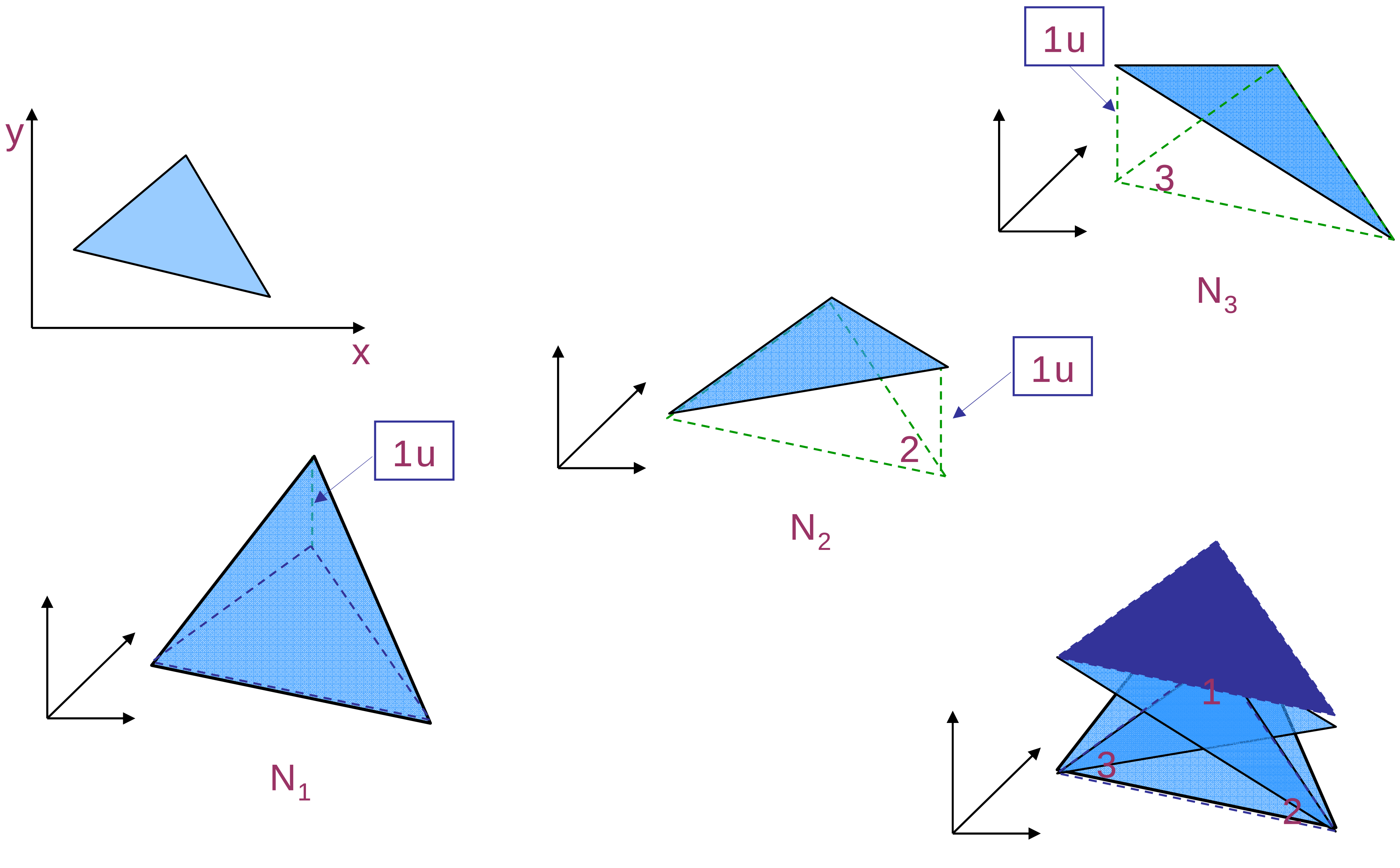
Si expresamos lo anterior matricialmente:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

La "**función de forma**" i_ésima del elemento es:

$$N_i = N_i(x,y) = \frac{1}{2a}\left[a_1 + b_1 x + c_1 y\right]$$

Interpretación geométrica de las funciones de forma:



CAMPO DE DEFORMACIONES EN EL ELEMENTO

Deformaciones:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_3}{\partial x} v_3$$

Es decir:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{N}_1}{\partial \boldsymbol{x}} & 0 & \frac{\partial \boldsymbol{N}_2}{\partial \boldsymbol{x}} & 0 & \frac{\partial \boldsymbol{N}_3}{\partial \boldsymbol{x}} & 0 \\ 0 & \frac{\partial \boldsymbol{N}_1}{\partial \boldsymbol{y}} & 0 & \frac{\partial \boldsymbol{N}_2}{\partial \boldsymbol{y}} & 0 & \frac{\partial \boldsymbol{N}_3}{\partial \boldsymbol{y}} \\ \frac{\partial \boldsymbol{N}_1}{\partial \boldsymbol{y}} & \frac{\partial \boldsymbol{N}_1}{\partial \boldsymbol{x}} & \frac{\partial \boldsymbol{N}_2}{\partial \boldsymbol{y}} & \frac{\partial \boldsymbol{N}_2}{\partial \boldsymbol{x}} & \frac{\partial \boldsymbol{N}_3}{\partial \boldsymbol{y}} & \frac{\partial \boldsymbol{N}_3}{\partial \boldsymbol{x}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \end{Bmatrix} = [\boldsymbol{B}]\{\boldsymbol{u}\}$$

La matriz [B] ¡¡¡es constante!!!

$$[B]=\frac{1}{2a}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$[B]=\begin{bmatrix} B_1 & \vdots & B_2 & \vdots & B_3 \end{bmatrix}$$

$$[B_i]=\begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} \end{bmatrix}$$

Campo de tensiones en el interior del elemento (Hipótesis de tensión plana)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ & \dfrac{E}{1-\nu^2} & 0 \\ \textit{Simétrica} & & \dfrac{E}{2(1+\nu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Campo de tensiones en el interior del elemento (Hipótesis de deformación plana)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ & 1 & 0 \\ \textbf{\textit{Simétrica}} & & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad [D] = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ & \dfrac{E}{1-\nu^2} & 0 \\ \textit{Simétrica} & & \dfrac{E}{2(1+\nu)} \end{bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\{\sigma\} = [D][B]\{u\}$$

$\{\sigma\}$ se denomina **vector tensión en el elemento**,

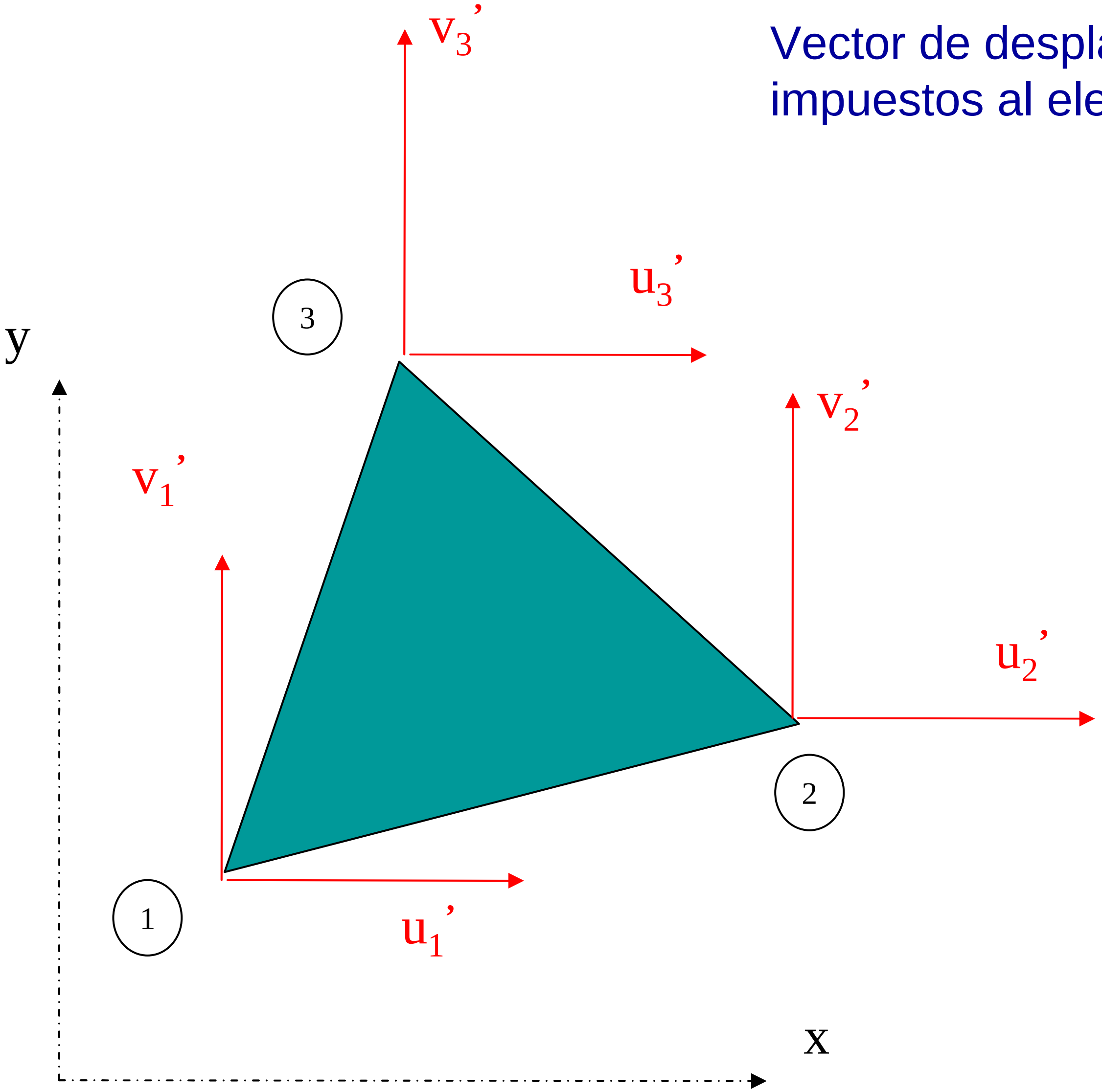
$\{u\}$ se denomina **vector desplazamientos nodales**,

$[D]$ es una matriz que depende de las propiedades elásticas del material

y, finalmente, $[B]$ es una matriz cuyos elementos dependen de las coordenadas cartesianas de los nudos del elemento.

Aplicación del P.T.V.

Vector de desplazamientos virtuales impuestos al elemento:



$$\{\boldsymbol{u}'\} = \begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \\ u_3' \\ v_3' \end{Bmatrix}$$

Campo de deformaciones virtuales:

$$\{\varepsilon'\} = [B]\{u'\}$$

$$\{F\}^T \{u'\} = Trabajo\ de\ las\ \ fuerzas\ nodales$$

$$\boxed{\{F\}^T \{u'\} = \int_{Volumen} \{\sigma\}^T \{\varepsilon'\}\, d\,Vol}$$

$$\{F\}^T\{u'\} = \int_{Volumen} \left(\{u\}^T [B]^T [D]^T\right) \cdot \left([B]\{u'\}\right) d\,Vol = \int_{Volumen} \{u\}^T [B]^T [D]^T [B]\{u'\}\, d\,Vol$$

Como la ecuación anterior debe verificarse para cualquier vector de desplazamientos virtuales {u'} impuesto al elemento:

$$\{F\}^T = \{u\}^T \left(\int_{Volumen} [B]^T [D]^T [B]\, d\,Vol \right)$$

$$\{F\}^T = \{u\}^T \left(\int_{Volumen} [B]^T [D]^T [B] \, d\,Vol \right)$$

ó, trasponiendo:

$$\{F\} = \left(\int_{Volumen} [B]^T [D] [B] \, d\,Vol \right) \{u\}$$

$$\{F\} = \left([B]^T [D] [B] Volumen \right) \{u\}$$

$$\boldsymbol{\{F\} = \left[K^e\right] \{u\}}$$

$$\left[K^e\right] = [B]^T [D] [B] Volumen$$

MATRIZ DE RIGIDEZ DEL ELEMENTO TRIANGULAR

$$\left[K^e\right] = \frac{e}{4a}\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & & & k_{44} & k_{45} & k_{46} \\ & sim. & & & k_{55} & k_{56} \\ & & & & & k_{66} \end{bmatrix}$$

$$k_{11} = d_{11}b_1^2 + d_{33}c_1^2$$
$$k_{22} = d_{22}c_1^2 + d_{33}b_1^2$$
$$k_{33} = d_{11}b_2^2 + d_{33}c_2^2$$
$$k_{44} = d_{22}c_2^2 + d_{33}b_2^2$$
$$k_{55} = d_{11}b_3^2 + d_{33}c_3^2$$
$$k_{66} = d_{22}c_3^2 + d_{33}b_3^2$$
$$k_{12} = \left(d_{12} + d_{33}\right)b_1c_1$$
$$k_{13} = d_{11}b_1b_2 + d_{33}c_1c_2$$
$$k_{14} = d_{12}b_1c_2 + d_{33}c_1b_2$$
$$k_{15} = d_{11}b_1b_3 + d_{33}c_1c_3$$
$$k_{16} = d_{12}b_1c_3 + d_{33}c_1b_3$$

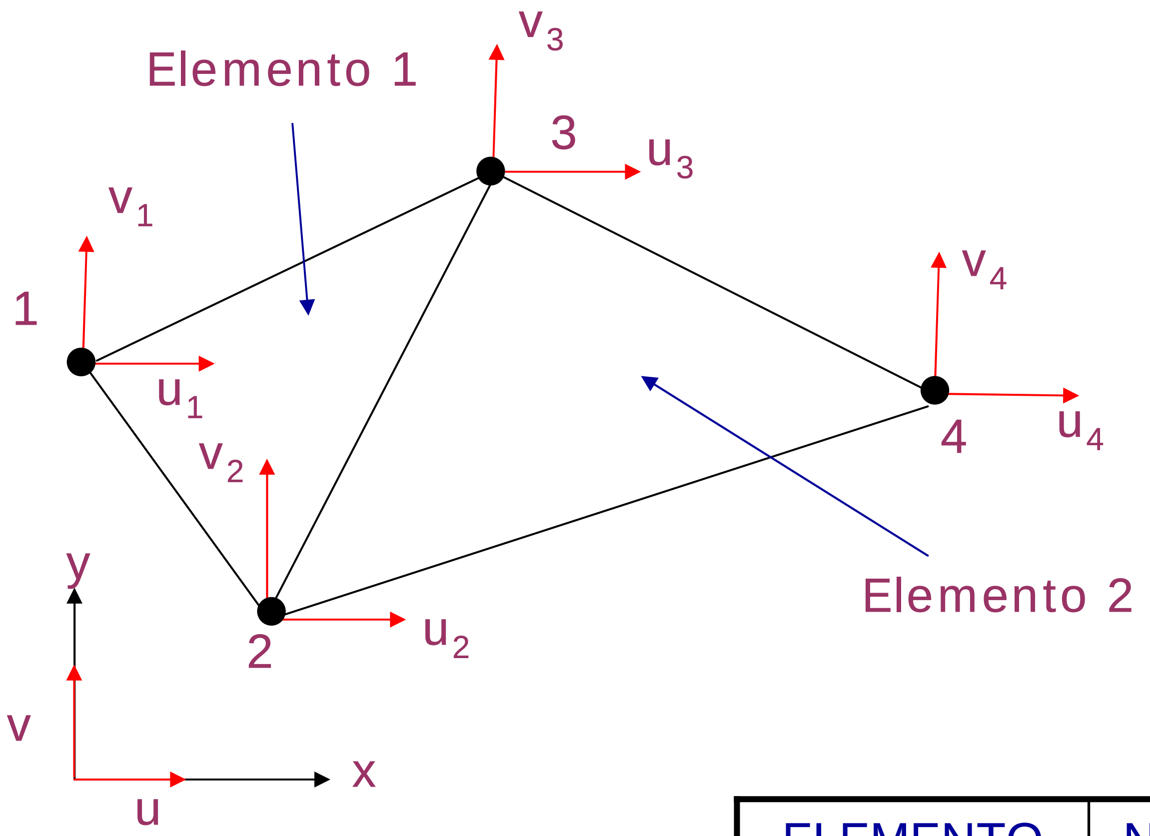
$$k_{23} = d_{12}b_2c_1 + d_{33}b_1c_2$$
$$k_{24} = d_{22}c_1c_2 + d_{33}b_1b_2$$
$$k_{25} = d_{12}b_3c_1 + d_{33}b_1c_3$$
$$k_{26} = d_{22}c_1c_3 + d_{33}b_1b_3$$
$$k_{34} = \left(d_{12} + d_{33}\right)b_2c_2$$
$$k_{35} = d_{11}b_2b_3 + d_{33}c_1c_3$$
$$k_{36} = d_{12}b_2c_3 + d_{33}b_3c_2$$
$$k_{45} = d_{21}b_3c_2 + d_{33}b_2c_3$$
$$k_{46} = d_{22}c_2c_3 + d_{33}b_3b_2$$
$$k_{56} = \left(d_{21} + d_{33}\right)b_3c_3$$

$$a = \frac{1}{2}\left(a_1 + a_2 + a_3\right)$$

e = espesor del elemento

ENSAMBLAJE

Se establece la relación entre los nodos de cada elemento y los nodos de la malla global



ELEMENTO	Nudo 1	Nudo 2	Nudo 3
1	1	2	3
2	2	4	3

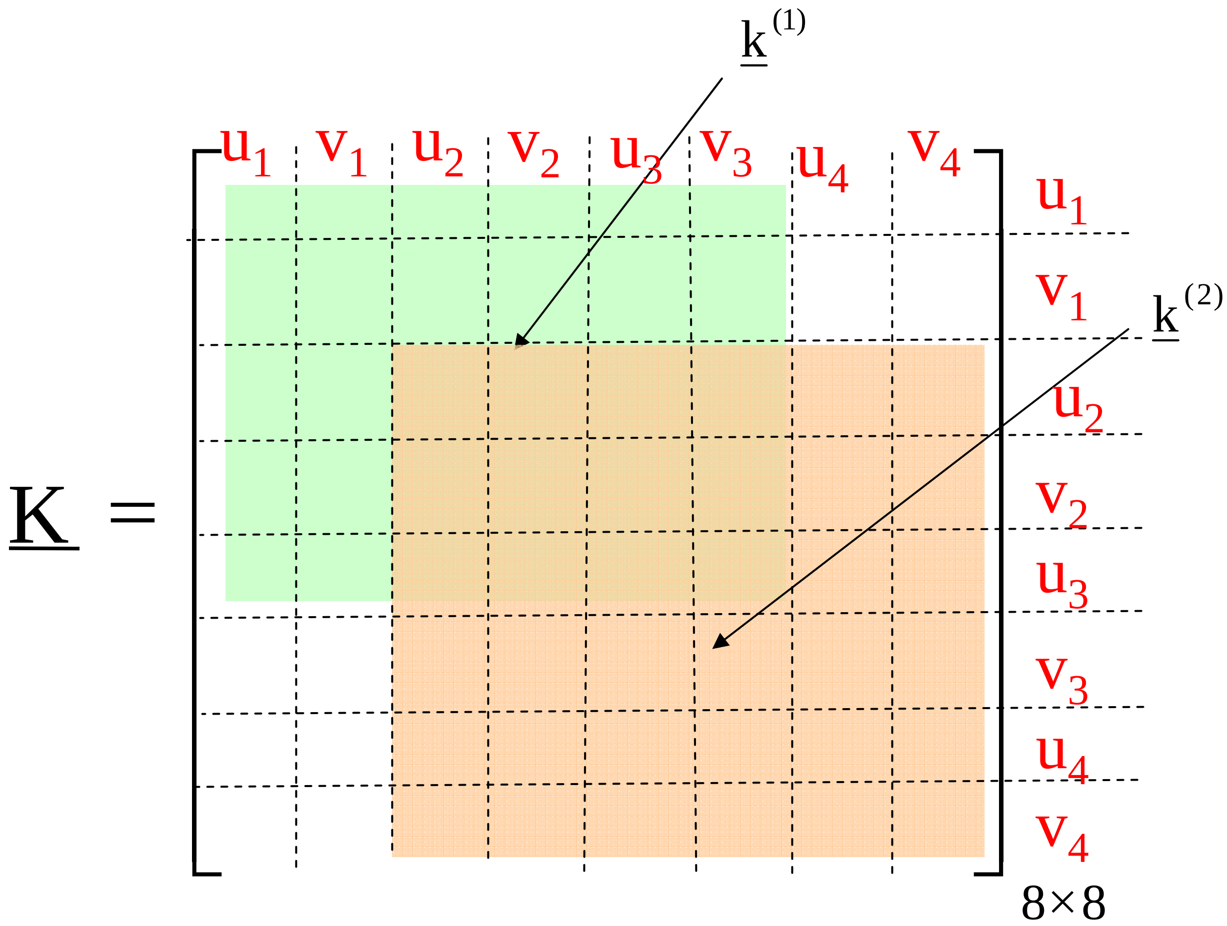
Matriz de rigidez del elemento 1

$$
\underline{k}^{(1)} = \begin{array}{cccccc|c}
u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & v_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & v_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & v_3
\end{array}
$$

Matriz de rigidez del elemento 2

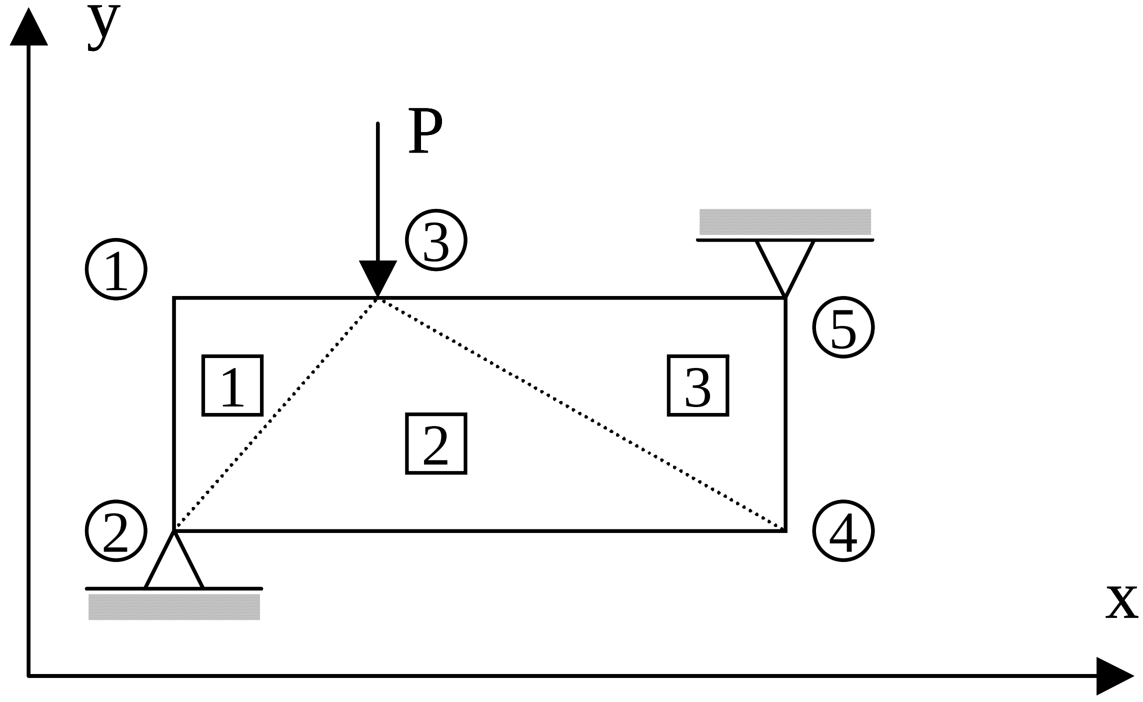
$$
\underline{k}^{(2)} = \begin{array}{cccccc|c}
u_2 & v_2 & u_3 & v_3 & u_4 & v_4 & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & v_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & v_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_4 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & v_4
\end{array}
$$

Cada elemento tiene 6 grados de libertad (2 por nudo)

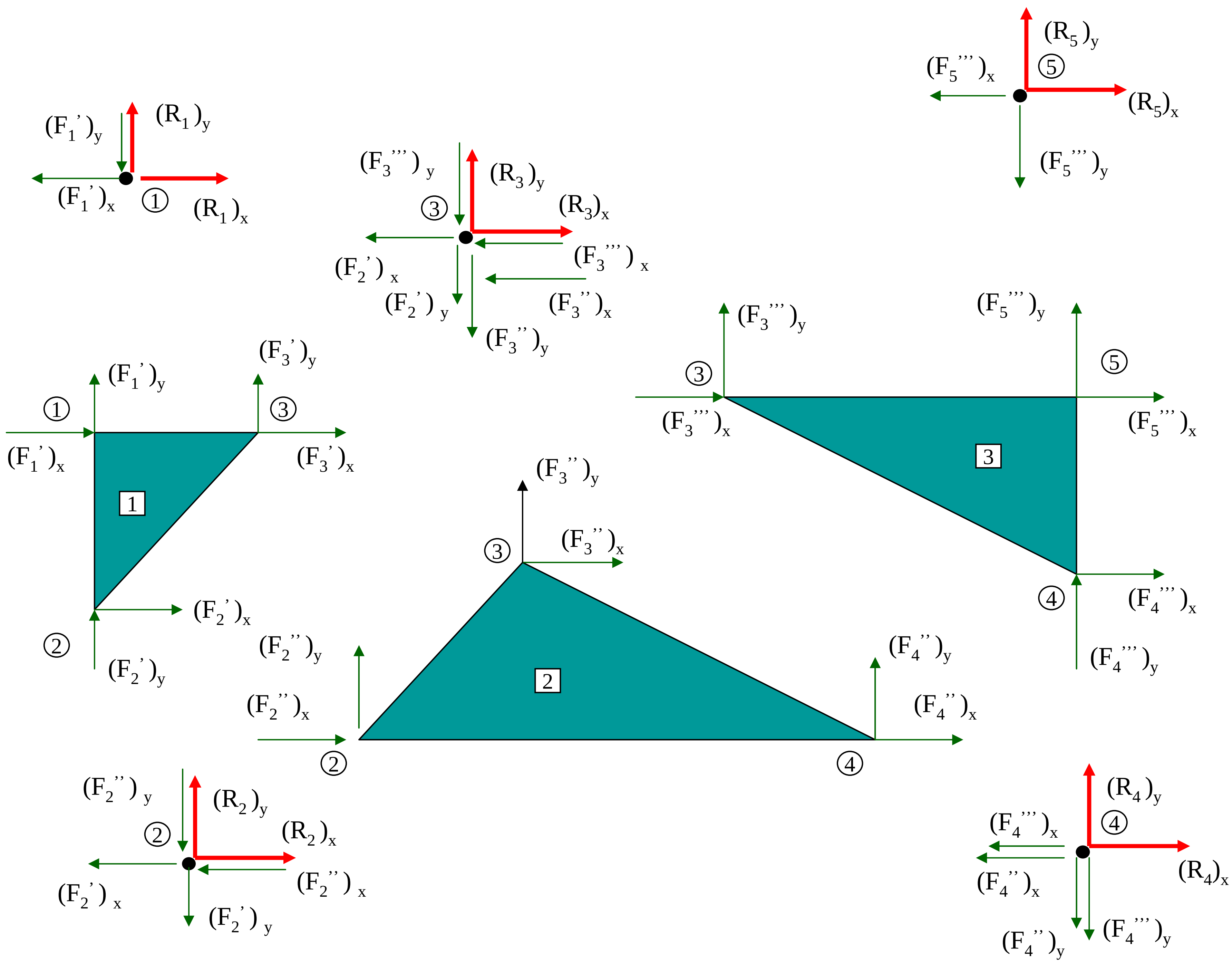


Las condiciones de contorno se incorporan como ya sabemos

EJEMPLO DE APLICACIÓN



Elemento 1 Nudos: 1 2 3
Elemento 2 Nudos: 2 4 3
Elemento 3 Nudos: 3 4 5



$(F_1')_y$
$(R_1)_y$
$(F_1')_x$
①
$(R_1)_x$
$(F_3''')_y$
$(R_3)_y$
③
$(R_3)_x$
$(F_2')_x$
$(F_3''')_x$
$(F_2')_y$
$(F_3'')_x$
$(F_3'')_y$
$(R_5)_y$
$(F_5''')_x$
⑤
$(R_5)_x$
$(F_5''')_y$
$(F_3')_y$
$(F_1')_y$
①
③
$(F_1')_x$
$(F_3')_x$
1
$(F_2')_x$
②
$(F_2')_y$
$(F_3''')_y$
$(F_5''')_y$
③
⑤
$(F_3''')_x$
$(F_5''')_x$
3
④
$(F_4''')_x$
$(F_4''')_y$
$(F_3'')_y$
③
$(F_3'')_x$
2
$(F_2'')_y$
$(F_2'')_x$
②
④
$(F_4'')_y$
$(F_4'')_x$
$(F_2'')_y$
$(R_2)_y$
②
$(R_2)_x$
$(F_2')_x$
$(F_2'')_x$
$(F_2')_y$
$(R_4)_y$
$(F_4''')_x$
④
$(F_4'')_x$
$(R_4)_x$
$(F_4'')_y$
$(F_4''')_y$

ELEMENTO 1

$$\{F^1\} = [K^e]^{elemento\,1}\ \{u^1\}$$

$$[K^e]^{elemento\,1} = \begin{bmatrix} k^1{}_{11} & k^1{}_{12} & k^1{}_{13} & k^1{}_{14} & k^1{}_{15} & k^1{}_{16} \\ & k^1{}_{22} & k^1{}_{23} & k^1{}_{24} & k^1{}_{25} & k^1{}_{26} \\ & & k^1{}_{33} & k^1{}_{34} & k^1{}_{35} & k^1{}_{36} \\ & & & k^1{}_{44} & k^1{}_{45} & k^1{}_{46} \\ & sim. & & & k^1{}_{55} & k^1{}_{56} \\ & & & & & k^1{}_{66} \end{bmatrix}$$

$$\{\boldsymbol{F}^1\} = \begin{Bmatrix} \left(\boldsymbol{F}_1'\right)_x \\ \left(\boldsymbol{F}_1'\right)_y \\ \left(\boldsymbol{F}_2'\right)_x \\ \left(\boldsymbol{F}_2'\right)_y \\ \left(\boldsymbol{F}_3'\right)_x \\ \left(\boldsymbol{F}_3'\right)_y \end{Bmatrix}$$

$$\{\boldsymbol{u}^1\} = \begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \end{Bmatrix}$$

ELEMENTO 2

$$\{F^2\} = [K^e]^{elemento\ 2}\{u^2\}$$

$$\{\boldsymbol{F}^2\} = \begin{Bmatrix} \left(\boldsymbol{F}_2''\right)_x \\ \left(\boldsymbol{F}_2''\right)_y \\ \left(\boldsymbol{F}_4''\right)_x \\ \left(\boldsymbol{F}_4''\right)_y \\ \left(\boldsymbol{F}_3''\right)_x \\ \left(\boldsymbol{F}_3''\right)_y \end{Bmatrix} \qquad \{\boldsymbol{u}^2\} = \begin{Bmatrix} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_4 \\ \boldsymbol{v}_4 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \end{Bmatrix}$$

ELEMENTO 3

$$\{F^3\} = [K^e]^{elemento\,3}\{u^3\}$$

$$\{\boldsymbol{F}^3\} = \begin{Bmatrix} (\boldsymbol{F}_3''')_x \\ (\boldsymbol{F}_3''')_y \\ (\boldsymbol{F}_4''')_x \\ (\boldsymbol{F}_4''')_y \\ (\boldsymbol{F}_5''')_x \\ (\boldsymbol{F}_5''')_y \end{Bmatrix} \qquad \{\boldsymbol{u}^3\} = \begin{Bmatrix} \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{v}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{v}_5 \end{Bmatrix}$$

ELEMENTO 1

$$\begin{Bmatrix} (F_1')_x \\ (F_1')_y \\ (F_2')_x \\ (F_2')_y \\ (F_3')_x \\ (F_3')_y \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 & 0 & 0 \\ & k_{22}^1 & k_{23}^1 & k_{24}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 & 0 & 0 \\ & & k_{33}^1 & k_{34}^1 & k_{35}^1 & k_{36}^1 & 0 & 0 & 0 & 0 \\ & & & k_{44}^1 & k_{45}^1 & k_{46}^1 & 0 & 0 & 0 & 0 \\ & & & & k_{55}^1 & k_{56}^1 & 0 & 0 & 0 & 0 \\ & & & & & k_{66}^1 & 0 & 0 & 0 & 0 \\ & Sim. & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 \\ & & & & & & & & & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

ELEMENTO 2

$$
\begin{Bmatrix} 0 \\ 0 \\ \left(F_2''\right)_x \\ \left(F_2''\right)_y \\ \left(F_3''\right)_x \\ \left(F_3''\right)_y \\ \left(F_4''\right)_x \\ \left(F_4''\right)_y \\ 0 \\ 0 \end{Bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & k_{11}^{2} & k_{12}^{2} & k_{13}^{2} & k_{14}^{2} & k_{15}^{2} & k_{16}^{2} & 0 & 0 \\
 & & & k_{22}^{2} & k_{23}^{2} & k_{24}^{2} & k_{25}^{2} & k_{26}^{2} & 0 & 0 \\
 & & & & k_{55}^{2} & k_{56}^{2} & k_{53}^{2} & k_{54}^{2} & 0 & 0 \\
 & & & & & k_{66}^{2} & k_{63}^{2} & k_{64}^{2} & 0 & 0 \\
 & Sim. & & & & & k_{33}^{2} & k_{34}^{2} & 0 & 0 \\
 & & & & & & & k_{44}^{2} & 0 & 0 \\
 & & & & & & & & 0 & 0 \\
 & & & & & & & & & 0
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{v}_4 \\ 0 \\ 0 \end{Bmatrix}
$$

ELEMENTO 3

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (F_3''')_x \\ (F_3''')_y \\ (F_4''')_x \\ (F_4''')_y \\ (F_5''')_x \\ (F_5''')_y \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & k_{11}^3 & k_{12}^3 & k_{13}^3 & k_{14}^3 & k_{15}^3 & k_{16}^3 \\ & & & & & k_{22}^3 & k_{23}^3 & k_{24}^3 & k_{25}^3 & k_{26}^3 \\ & Sim. & & & & & k_{33}^3 & k_{34}^3 & k_{35}^3 & k_{36}^3 \\ & & & & & & & k_{44}^3 & k_{45}^3 & k_{46}^3 \\ & & & & & & & & k_{55}^3 & k_{56}^3 \\ & & & & & & & & & k_{66}^3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{v}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{v}_5 \end{Bmatrix}$$

MATRIZ DE RIGIDEZ DE LA ESTRUCTURA

$$[\boldsymbol{K}]=\begin{bmatrix}
\boldsymbol{k}_{11}^{1} & \boldsymbol{k}_{12}^{1} & \boldsymbol{k}_{13}^{1} & \boldsymbol{k}_{14}^{1} & \boldsymbol{k}_{15}^{1} & \boldsymbol{k}_{16}^{1} & 0 & 0 & 0 & 0 \\
 & \boldsymbol{k}_{22}^{1} & \boldsymbol{k}_{23}^{1} & \boldsymbol{k}_{24}^{1} & \boldsymbol{k}_{25}^{1} & \boldsymbol{k}_{26}^{1} & 0 & 0 & 0 & 0 \\
 & & \boldsymbol{k}_{33}^{1}+\boldsymbol{k}_{11}^{2} & \boldsymbol{k}_{34}^{1}+\boldsymbol{k}_{12}^{2} & \boldsymbol{k}_{35}^{1}+\boldsymbol{k}_{15}^{2} & \boldsymbol{k}_{36}^{1}+\boldsymbol{k}_{16}^{2} & \boldsymbol{k}_{13}^{2} & \boldsymbol{k}_{14}^{2} & 0 & 0 \\
 & & & \boldsymbol{k}_{44}^{1}+\boldsymbol{k}_{22}^{2} & \boldsymbol{k}_{45}^{1}+\boldsymbol{k}_{25}^{2} & \boldsymbol{k}_{46}^{1}+\boldsymbol{k}_{26}^{2} & \boldsymbol{k}_{23}^{2} & \boldsymbol{k}_{24}^{2} & 0 & 0 \\
 & & & & \boldsymbol{k}_{55}^{1}+\boldsymbol{k}_{55}^{2}+\boldsymbol{k}_{11}^{3} & \boldsymbol{k}_{56}^{1}+\boldsymbol{k}_{56}^{2}+\boldsymbol{k}_{12}^{3} & \boldsymbol{k}_{53}^{2}+\boldsymbol{k}_{13}^{3} & \boldsymbol{k}_{54}^{2}+\boldsymbol{k}_{14}^{3} & \boldsymbol{k}_{15}^{3} & \boldsymbol{k}_{16}^{3} \\
 & & & & & \boldsymbol{k}_{66}^{1}+\boldsymbol{k}_{66}^{2}+\boldsymbol{k}_{22}^{3} & \boldsymbol{k}_{63}^{2}+\boldsymbol{k}_{23}^{3} & \boldsymbol{k}_{64}^{2}+\boldsymbol{k}_{24}^{3} & \boldsymbol{k}_{25}^{3} & \boldsymbol{k}_{26}^{3} \\
 & \boldsymbol{Sim.} & & & & & \boldsymbol{k}_{33}^{2}+\boldsymbol{k}_{33}^{3} & \boldsymbol{k}_{34}^{2}+\boldsymbol{k}_{34}^{3} & \boldsymbol{k}_{35}^{3} & \boldsymbol{k}_{36}^{3} \\
 & & & & & & & \boldsymbol{k}_{44}^{2}+\boldsymbol{k}_{44}^{3} & \boldsymbol{k}_{45}^{3} & \boldsymbol{k}_{46}^{3} \\
 & & & & & & & & \boldsymbol{k}_{55}^{3} & \boldsymbol{k}_{56}^{3} \\
 & & & & & & & & & \boldsymbol{k}_{66}^{3}
\end{bmatrix}$$

$$\{R\} = [K]\{u\}$$

$$\{R\} = \begin{Bmatrix} (R_1)_x \\ (R_1)_y \\ (R_2)_x \\ (R_2)_y \\ (R_3)_x \\ (R_3)_y \\ (R_4)_x \\ (R_4)_y \\ (R_5)_x \\ (R_5)_y \end{Bmatrix} \begin{Bmatrix} (F_1')_x \\ (F_1')_y \\ (F_2')_x + (F_2'')_x \\ (F_2')_y + (F_2'')_y \\ (F_3')_x + (F_3'')_x + (F_3''')_x \\ (F_3')_y + (F_3'')_y + (F_3''')_y \\ (F_4'')_x + (F_4''')_x \\ (F_4'')_y + (F_4''')_y \\ (F_5''')_x \\ (F_5''')_y \end{Bmatrix} \qquad \{\boldsymbol{u}\} = \begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{v}_4 \\ \boldsymbol{u}_5 \\ \boldsymbol{v}_5 \end{Bmatrix}$$

$$(R_1)_x = (R_1)_y = (R_3)_x = (R_4)_x = (R_4)_y = 0 \quad y \quad (R_3)_y = -P$$

$$\{F\} = \begin{Bmatrix} 0 \\ 0 \\ (R_2)_x \\ (R_2)_y \\ 0 \\ -P \\ 0 \\ 0 \\ (R_5)_x \\ (R_5)_y \end{Bmatrix}$$

Incógnitas: 4

$$u_2 = v_2 = u_5 = v_5 = 0$$

$$\{u\} = \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ 0 \\ 0 \end{Bmatrix}$$

Incógnitas: 6

$$
\begin{Bmatrix} 0 \\ 0 \\ (\boldsymbol{R}_2)_x \\ (\boldsymbol{R}_2)_y \\ 0 \\ -\boldsymbol{P} \\ 0 \\ 0 \\ (\boldsymbol{R}_5)_x \\ (\boldsymbol{R}_5)_y \end{Bmatrix}
=
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & k_{15}^1 & k_{16}^1 & 0 & 0 & 0 & 0 \\
 & k_{22}^1 & k_{23}^1 & k_{24}^1 & k_{25}^1 & k_{26}^1 & 0 & 0 & 0 & 0 \\
 & & k_{33}^1+k_{11}^2 & k_{34}^1+k_{12}^2 & k_{35}^1+k_{15}^2 & k_{36}^1+k_{16}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 \\
 & & & k_{44}^1+k_{22}^2 & k_{45}^1+k_{25}^2 & k_{46}^1+k_{26}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 \\
 & & & & k_{55}^1+k_{55}^2+k_{11}^3 & k_{56}^1+k_{56}^2+k_{12}^3 & k_{53}^2+k_{13}^3 & k_{54}^2+k_{14}^3 & k_{15}^3 & k_{16}^3 \\
 & & & & & k_{66}^1+k_{66}^2+k_{22}^3 & k_{63}^2+k_{23}^3 & k_{64}^2+k_{24}^3 & k_{25}^3 & k_{26}^3 \\
 & Sim & & & & & k_{33}^2+k_{33}^3 & k_{34}^2+k_{34}^3 & k_{35}^3 & k_{36}^3 \\
 & & & & & & & k_{44}^2+k_{44}^3 & k_{45}^3 & k_{46}^3 \\
 & & & & & & & & k_{55}^3 & k_{56}^3 \\
 & & & & & & & & & k_{66}^3
\end{bmatrix}
\begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \\ 0 \\ 0 \\ \boldsymbol{u}_3 \\ \boldsymbol{v}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{v}_4 \\ 0 \\ 0 \end{Bmatrix}
$$

¡10 ECUACIONES CON 10 INCÓGNITAS!

“CONDICIONES DE USO” DEL ELEMENTO TRIANGULAR DE TURNER

- Problemas en los que no haya flexiones

  Porque a los nodos del elemento solo se les ha posibilitado moverse en su plano

- Problemas en los que la deformación pueda aceptarse como constante

  Porque con las funciones de forma definidas la deformación en el elemento es constante

- Problemas en los que la tensión pueda aceptarse como constante

  Porque con las funciones de forma definidas la tensión en el elemento es constante

Ejemplos de problemas en los que puede aplicarse este elemento:

1. Tuberías
2. Tanques cilíndricos
3. Tanques esféricos,...