Continuity

A cont. function: small changes in input $\Rightarrow$ small changes in output.

DEF.

Let f be a function defined on $(x_0 - p, x_0 + p)$, $p > 0$

$$f \text{ is } \textbf{continuous} \text{ at } x_0 \iff \lim_{x \longrightarrow x_0} f(x) = f(x_0).$$

f MUST be defined at x_0.

If f is not continuous $\longrightarrow$ **discontinuous** at x_0:

- $\lim_{x \longrightarrow x_0} f(x)$ does not exist, or
- the limit exist but is not equal to $f(x_0)$

- A function $f(x)$ is continuous on $\mathbb{R}$ if it is continuous $\forall x \in \mathbb{R}$.
- $f(x)$ is continuous on (a, b) if it is continuous at each point in (a, b).
- $f(x)$ is continuous on $[a, b]$ if it is continuous on (a, b) and

$$f(a) = \lim_{x \longrightarrow a^+} f(x), \; f(b) = \lim_{x \longrightarrow b^-} f(x).$$

Basic properties

Let f, g be continuous at x_0, then the following functions are continuous at x_0

- $\alpha f + \beta g$
- fg
- $1/f$ if $f(x_0) \neq 0$.
- Composite function: if g cont. at x_0 and f cont. at $g(x_0)$, then $f \circ g$ is continuous at x_0.

Some continuous functions

The following functions are continuous on their domains

- polynomials: $p(x)$
- rational functions: $p(x)/q(x)$
- trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\arcsin(x), \dots$
- hyperbolic functions $\sinh(x)$, $\cosh(x), \dots$
- $\exp(x)$, $\ln(x)$ and $\sqrt[n]{x}$

Theorem (Intermediate Value Theorem)

If $f(x)$ is a continuous function on $[a, b]$ and K is a number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = K$.

Theorem (Bolzano's Theorem)

If $f(x)$ is a continuous function on $[a, b]$ and $f(a) \cdot f(b) < 0$, then there is a $c \in (a, b)$ such that $f(c) = 0$.

A function f is **bounded** if the set of its values is bounded. That is, if there exists a number $M > 0$ such that $|f(x)| \leq M$, for all x in its domain.

Theorem (Extreme Value Theorem)

If $f(x)$ is a continuous function on the closed interval $[a, b]$, then f attains its maximum and minimum value. That is, there exist numbers x_m and x_M in $[a, b]$ such that:
$f(x_M) \geq f(x) \geq f(x_m) \quad$ *for all* $x \in [a, b]$.