Derivatives

DEF.

A function f is **differentiable** at $x \iff$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is a finite number.
If f is differentiable then $f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \longrightarrow$ **derivative** of f at x.
$f' \longrightarrow$ new function.

DEF. (ALTERNATIVE DEF.)

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The tangent line to $f(x)$ at x_0: $y = f'(x_0)(x - x_0) + f(x_0)$.

Properties

1. $(c_1 f + c_2 g)' = c_1 f' + c_2 g', \quad c_1, c_2 \in \mathbb{R}$
2. $(f \cdot g)' = f'g + fg'$
3. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

Theorem (The Chain Rule)

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and verifies

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Basic derivatives

1. $c' = 0$
2. $(x^n)' = nx^{n-1}$
3. $(e^x)' = e^x, \qquad (\log x)' = \dfrac{1}{x}$
4. $(\sin\ x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad (\tan x)' = \dfrac{1}{\cos^2 x}$
5. $(\arctan x)' = \dfrac{1}{1+x^2}, \qquad (\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$
6. $(\sinh x)' = \cosh, \qquad (\cosh x) = \sinh x$

Theorem

f differentiable $\Rightarrow$ f continuous

Theorem (Rolle's Theorem)

Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b)$, then there is at least a number $c \in (a, b)$ such that

$$f'(c) = 0$$

Theorem (Mean Value Theorem)

Let f be differentiable on (a, b) and continuous on $[a, b]$, then there is at least a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or, equivalently $f(b) - f(a) = f'(c)(b - a)$

Theorem (L'Hôpital's Rule)

Let f and g be differentiable functions on (a, b), except possibly at the point $x_0 \in (a, b)$. If $\lim_{x \to x_0} \dfrac{f(x)}{g(x)}$ is the indeterminate form $\dfrac{0}{0}$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},$$

whenever $\lim_{x \to x_0} \dfrac{f'(x)}{g'(x)}$ exists or it is infinite.

Extensions. L'Hôpital's Rule can be applied also in the following cases:

- If the indeterminate form is $\frac{\infty}{\infty}$ with all the possible signs.
- If the limit is taken when $x_0 \to \pm\infty$
- To one-sided limits

- **Implicit differentiation:**

  $F(x, y) = 0,$

  $\frac{d}{dx}$ both sides of the equation $\rightarrow \frac{dy}{dx}$

- **Higher order derivatives:**

$$\frac{d^2 f}{dx^2} = f''(x), \qquad \frac{d^3 f}{dx^3} = f'''(x), \cdots, \frac{d^n f}{dx^n} = f^{(n)}(x)$$