Real Sequences

Def.

A **real sequence** $\{a_n\}$ is a map:

$$a_n : \mathbb{N} \to \mathbb{R}.$$

$a_1, a_2, a_3 \ldots \to$ terms of the sequence.
$a_n \to$ general term.
Also with $n = 0$: $a_0, a_1, a_2 \ldots$.

Def.

A sequence $\{a_n\}$ is **convergent** if $\lim_{n\to\infty} a_n = L$, for L finite.

The **limit** of a sequence $\{a_n\}$ is L if for every $\epsilon > 0 \; \exists \, N \in \mathbb{N}$ such that if $n > N \Rightarrow |a_n - L| < \varepsilon$. There is an alternative def. for L infinite.

If the sequence is not convergent $\to$ divergent.
Properties of limits of sequences are the same of functions.

To compute limits

- Use the concept of the **limit of a function:**

  Let $f(x)$ be a function and $\{a_n\}$ the sequence $f(n) = a_n$.

  If $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} a_n = L$

  $\longrightarrow$ All the tools of limits of functions, such as L'Hopital Rule.

- The **Sandwich lemma** of sequences:

  If $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$ (finite or infinite)

  and $\{c_n\}$ verifies $a_n \leq c_n \leq b_n, \quad \forall\, n \in \mathbb{N}$,

  then $\lim_{n \to \infty} c_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$

DEF. (A SEQUENCE $\{a_n\}$ IS:)

1. bounded from above if $\exists\, C \in \mathbb{R}$ such that $a_n \leq C$.
2. bounded from below if $\exists\, C \in \mathbb{R}$ such that $a_n \geq C$.
3. **bounded** if it is bounded from above and below ($\exists\, C_1, C_2 \in \mathbb{R}$, s. t. $C_1 \leq a_n \leq C_2$).

DEF. (A SEQUENCE $\{a_n\}$ IS)

1. monotonically increasing if $a_n < a_{n+1}$ (non decreasing if $a_n \leq a_{n+1}$).
2. monotonically decreasing if $a_n > a_{n+1}$ (non increasing if $a_n \geq a_{n+1}$).
3. **monotonic** if it is one of the previous cases.

Theorem

$\{a_n\}$ monotonic and bounded $\Rightarrow$ $\{a_n\}$ convergent.

Theorem (Stolz Test)

If $\{a_n\}$ and $\{b_n\}$ verify one of the following:

1. *$\{b_n\}$ is monotonically increasing with $\lim_{n\to\infty} b_n = \infty$,*
2. *$\{b_n\}$ is monotonically decreasing, with $b_n \neq 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.*

Whenever $\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$, exists for L finite or infinite, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}.$$

Theorem (Stirling's Formula)

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$