Series

Def. (Series)

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots .$$

$S_n = a_1 + a_2 + a_3 + \cdots a_n \to$ partial sum of n terms is
If $\lim_{n\to\infty} S_n = S < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ **converges**.

$$S = \lim_{n\to\infty} S_n = a_1 + a_2 + a_3 + a_4 + \cdots .$$

Otherwise $\to$ the series diverges

Properties

1. $\sum a_n, \sum b_n$ conv $\Rightarrow \sum (c_1 a_n + c_2 b_n) = c_1 \sum a_n + c_2 \sum b_n$ conv
2. $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ div
3. $\sum a_n$ conv $\Rightarrow \lim_{n\to\infty} a_n = 0$. (But $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum a_n$ conv)

Theorem (The geometrical sum)

Converges if $0 < |r| < 1 \rightarrow$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Theorem (The telescoping series. $(a_n = b_n - b_{n+1})$

$$\sum_{n=1}^{\infty}(b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots$$

$\rightarrow S_n = b_1 - b_{n+1}.$

This series converges $\iff \lim_{n\to\infty} b_n < \infty$ *and* $S = b_1 - \lim_{n\to\infty} b_n$

Theorem (The p-series. $p = 1 \rightarrow$ harmonic series)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

1. *converges if* $p > 1$
2. *diverges if* $0 < p \leq 1$

CONVERGENCE TEST FOR SERIES

1. **Direct comparison test:** $\{a_n\}$ and $\{b_n\}$ $\to$ positive terms

$$0 < a_n \leq b_n,\ \forall\, n \quad \longrightarrow \quad \begin{array}{l} \sum b_n \text{ conv} \Rightarrow \sum a_n \text{ conv} \\ \sum a_n \text{ div} \Rightarrow \sum b_n \text{ div} \end{array}$$

2. **Limit comparison test:** $\{a_n\}$ and $\{b_n\}$ $\to$ positive terms

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \quad L \text{ finite and positive}$$

$$\Downarrow$$

$$\sum a_n \text{ and } \sum b_n \text{ have the same behaviour}$$

both converge or both diverge

3. **Root test:** $\{a_n\} \to$ positive terms

$$\lim_{n\to\infty} \sqrt[n]{a_n} < 1 \Rightarrow \sum a_n \text{ conv}$$
$$\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \Rightarrow \sum a_n \text{ div}$$
$$(\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \text{ the test does not conclude})$$

4. **Quotient test:** $\{a_n\} \to$ positive terms

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n \text{ conv}$$
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \sum a_n \text{ div}$$
$$(\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ the test does not conclude})$$

5. **Leibniz test for alternating series:** $\{a_n\} \to$ positive terms

$$\text{If } a_{n+1} \leq a_n \text{ and } \lim_{n\to\infty} a_n = 0$$
$$\Downarrow$$
$$\text{The alternating series } \sum(-1)^n a_n \text{ converges conditionally}$$
$$\left(\sum(-1)^{n+1} a_n\right)$$

DEF.

AC. $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent

CC. $\sum a_n$ conv but $\sum |a_n|$ div then $\sum a_n$ **conditionally convergent**

Absolute convergence	$\Longrightarrow$	Conditional convergence
No conditional convergence	$\Longrightarrow$	No absolute convergence

Error. Alternating series

$$S = S_N + R_N = \sum_{n=1}^{N} (-1^n) a_n + R_N \quad \Rightarrow \; |R_N| \leq a_{N+1}$$

Note. We can differentiate or integrate an infinite series to obtain another series.

Power Series

Def.

A **power series** at x_0 is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

Theorem (Convergence of a power series)

A power series at x_0 verifies only one of the following:

1. *The series converges only at x_0.*
2. *There is a real number $\rho > 0$ such that the series is*
   - *absolutely convergent for $|x - c| < \rho$*
   - *divergent for $|x - c| > \rho$*
3. *The series is absolutely convergent for every $x \in \mathbb{R}$*

Radius of convergence: ρ ($\rho = 0$, $\rho < \infty$ or $\rho = \infty$)

- $\dfrac{1}{\rho} = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$
- $\dfrac{1}{\rho} = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$, if this limit exists

Interval of convergence: the set of all x for which the series converges

THEOREM

If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has $\rho > 0 \Rightarrow$
$f(x)$ is continuous, differentiable and integrable on $(x_0 - \rho, x_0 + \rho)$.

The derivative and the integral $\to$ computed term by term.
Same radius as f. (The interval of convergence may be different)

Properties. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$.
2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{Nn}$.
3. $c_1 f(x) + c_2 g(x) = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) x^n$.

Def.

If f has all the derivatives at x_0,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is the **Taylor series** of f at x_0
(for $x_0 = 0$ also called the Mac Laurin series of f)

Theorem

If f has all the derivatives on an open interval I containing x_0 then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

$\iff \exists\, \xi$ *between x and x_0 such that*

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} = 0, \quad \forall\, x \in I.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \cdots, \quad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdots, \quad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \cdots, \quad -\infty < x < \infty$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)} \cdots, \quad -1 \leq x \leq 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n \cdots, \quad -1 < x < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} \cdots, \quad -1 < x \leq 1$$