

# INTEGRATION TECHNIQUES

## THE DEFINITE INTEGRAL

$$\int_a^b f(x)$$

Is the **area under the graph** of the function  $f(x) \geq 0$ , over the  $x$ -axis on  $x \in [a, b]$ .

To compute the definite integral  $\rightarrow$  divide the interval into  $n$  subintervals and approximate the function by a constant.

Then, compute the area of  $n$  rectangles. If  $f \simeq f(x_i)$  at the  $i$ -th interval, with all the intervals with same length,  $\Delta x$ , then

$$\int_a^b f(x) \simeq f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

**Diff.:** Given  $F(x)$ , find  $f(x) \leftrightarrow$  **Integration:** Given  $f(x)$ , find  $F(x)$

$$\frac{dF(x)}{dx} = f(x).$$

A function  $F(x)$  solving the second problem is called an **antiderivative primitive** or an **indefinite integral** of  $f(x)$ .

# PROPERTIES OF THE INTEGRAL

$$\textcircled{1} \int_a^b c_1 f + c_2 g =$$

$$c_1 \int_a^b f + c_2 \int_a^b g$$

$$\textcircled{2} \int_a^b f = \int_a^c f + \int_c^b f$$

$$\textcircled{3} \int_a^b f = - \int_b^a f$$

$$\textcircled{4} \int_a^a f = 0$$

$$\textcircled{5} \int_a^b fg \neq \int_a^b f \int_a^b g$$

$$\textcircled{6} f \geq g \Rightarrow \int_a^b f \geq \int_a^b g$$

$$\textcircled{7} f \geq 0 \Rightarrow \int_a^b f \geq 0$$

$$\text{if } f \leq 0 \Rightarrow \int_a^b f \leq 0$$

$$\textcircled{8} \left| \int_a^b f \right| \leq \int_a^b |f|$$

$$\textcircled{9} m \leq f(x) \leq M, \forall x \in [a, b] \Rightarrow \\ m(b-a) \leq \int_a^b f(x) \leq M(b-a)$$

## BASIC ANTIDERIVATIVES

$$\int x^n = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int e^{ax} = \frac{1}{a}e^{ax} + c$$

$$\int \sin x = -\cos x + c$$

$$\int \cos x = \sin x + c$$

$$\int \frac{1}{\cos^2 x} = \tan x + c$$

$$\int \frac{1}{\sin^2 x} = -\cot x + c$$

$$\int \frac{1}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \sinh x = \cosh x + c$$

$$\int \cosh x = \sinh x + c$$

# INTEGRATION TECHNIQUES

## INTEGRATION BY CHANGE OF VARIABLES (CV)

- Definite integral:  $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt$

- Indefinite integral:  $\int f(x)dx = \int f(g(t))g'(t)dt$

→ undo the change

## INTEGRATION BY PARTS (IBP): $\int u dv = uv - \int v du$

- Definite integral:  $\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)'g(x)dx$

- Indefinite integral:  $\int f(x)g'(x)dx = f(x)g(x) - \int f(x)'g(x)dx$

# RATIONAL FUNCTIONS: PARTIAL FRACTION DECOMPOSITION

$$\int \frac{P(x)}{Q(x)} dx \rightarrow P, Q \text{ polynomials}$$

- If  $\deg(P) \geq \deg(Q) \rightarrow$  **divide the polynomials:**  
 $P(x) = Q(x)C(x) + R(x) \rightarrow$

$$\int \frac{P(x)}{Q(x)} dx = \int C(x) + \int \frac{R(x)}{Q(x)} dx.$$

- $\int \frac{R(x)}{Q(x)} dx$  with  $\deg(R(x)) < \deg(Q(x))$ :

I) First, check that the integral is not immediate:

ln type  $\rightarrow \int \frac{2x+3}{x^2+3x+8} dx = \ln|x^2+3x+8| + c.$

arctan type  $\rightarrow \int \frac{dx}{x^2+8} = \frac{1}{\sqrt{8}} \arctan \frac{x}{\sqrt{8}} + c.$

II) If not  $\rightarrow$  **Do partial fraction decomposition.**

# RATIONAL FUNCTIONS: PARTIAL FRACTION DECOMPOSITION

Factor in denominator	Term in partial fraction decomposition
$x - b$	$\frac{A}{x - b}$
$(x - b)^k$	$\frac{A_1}{x - b} + \frac{A_2}{(x - b)^2} + \cdots + \frac{A_k}{(x - b)^k}, \quad k = 1, 2, 3, \dots$
$(x - a)^2 + b^2$	$\frac{Ax + B}{(x - a)^2 + b^2}$
$\left((x - a)^2 + b^2\right)^k$	$\frac{A_1x + B_1}{(x - a)^2 + b^2} + \cdots + \frac{A_kx + B_k}{\left((x - a)^2 + b^2\right)^k}, \quad k = 1, 2, 3, \dots$

For each factor in the denominator add the corresponding term of the table and compute the unknowns ( $A, B, A_1, B_1, A_2, B_2, \dots$ ) by setting equal denominators. After, compute the integrals of each term.

# IRRATIONAL FUNCTIONS OR INTEGRALS INVOLVING ROOTS

Do a change of variables that eliminates the roots.

$$\int R \left[ \left( \frac{ax+b}{cx+d} \right)^{p_1/q_1}, \dots, \left( \frac{ax+b}{cx+d} \right)^{p_r/q_r} \right] \rightarrow$$
$$t^m = \frac{ax+b}{cx+d}, \quad m = \text{lcm}(q_1, \dots, q_r).$$

$R = \frac{P}{Q}$  is a rational function of its variables,  $P$ ,  $Q$  are polynomials.

$\text{lcm} \rightarrow$  least common multiple.

# INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

- $\int \sin^{2n} x, \int \cos^{2n} x \rightarrow$   
double angle formulas:  $\cos 2x = \cos^2 x - \sin^2 x$
- $\int \sin^{2n+1} x = \int \sin^{2n} x \sin x = \int (1 - \cos^2 x)^n \sin x$
- $\int \cos^{2n+1} x = \int \cos^{2n} x \cos x = \int (1 - \sin^2 x)^n \cos x$
- $\int \sin mx \cos nx \rightarrow$  trig formulas
- $\int R(\sin x, \cos x) \rightarrow$   
R odd in  $\sin x \rightarrow t = \cos x$   
R odd in  $\cos x \rightarrow t = \sin x$   
R even in  $\cos x$  and  $\sin x \rightarrow t = \tan x$   
Rest of problems  $\rightarrow t = \tan x/2,$   
$$\left( \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt \right)$$

# SOME CHANGE OF VARIABLES

$$\textcircled{1} \int R(x, \sqrt{x^2 + a^2}) \rightarrow x = a \tan t$$

$$\textcircled{2} \int R(x, \sqrt{x^2 - a^2}) \rightarrow x = \frac{a}{\cos t}$$

$$\textcircled{3} \int R(x, \sqrt{a^2 - x^2}) \rightarrow x = a \sin t$$