Unit 1

Functions of a real variable

1.1 The real line

The real numbers are ordered like points on a line.

The positive whole numbers $\{1, 2, 3, 4, \ldots\}$ that arise from the counting process are called the Natural numbers (zero can be considered or not as a natural number, we will not do it).

The arithmetic operations of addition and multiplication can be performed within the natural numbers but substraction and division lead to the introduction of

Zero: $3 - 3 = 0$

Negative numbers: $2 - 6 = -4$

Fractions: $3 \div 5 = \dfrac{3}{5}$

So, we can classify numbers in the following way:

- **Natural numbers:** $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- **Integers:** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$
  zero and the positive and negative whole numbers.
- **Rational numbers:** $\mathbb{Q} = \left\{\dfrac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\right\}$.
- **Irrational numbers:** real numbers that are not rational such as $\sqrt{2}, \pi, \mathrm{e}$.
- **Real numbers:** $\mathbb{R} = \text{Irrational} + \text{Rational numbers}$.

All these numbers also arise as solution of equations:
$x - 1 = 0 \to x = 1$,
$x + 3 = 0 \to x = -3$,
$2x + 1 = 0 \to x = -\frac{1}{2}$,
$x^2 = 2 \to x = \pm\sqrt{2}$,
$x^2 + 1 = 0 \to x = \pm\sqrt{-1} \notin \mathbb{R} \to$ They are the **Complex numbers:**
$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$.

MATHEMATICAL PROOFS

The definition of the Natural numbers is the following:

$1 \in \mathbb{N}$, if $n \in \mathbb{N}$, so does $n + 1$

Let us see how this definition works:
$3 \in \mathbb{N} \Rightarrow 3 + 1 = 4 \in \mathbb{N}$,
$3/2 \notin \mathbb{N} \Rightarrow 3/2 + 1 = 5/2 \notin \mathbb{N}$.

This definition of the natural numbers introduce as the process of induction:

Proof by induction

Is a technique for proving a statement that is asserted about every natural number:

- The statement is true for $n = 1$
- If the statement is true for $n \in \mathbb{N}$, then it is also true for its successor: $n+1 \in \mathbb{N}$

This implies that the statement is true for every natural number n.

For instance, we can prove by induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Direct proof

The idea is to prove the statement directly

For example, to prove that $(a - b)(a + b) = a^2 - b^2$,
we just have to compute directly $(a - b)(a + b) = a^2 - ba + ab - b^2 = a^2 - b^2$.

Proof by contradiction or Reductio ad absurdum

- You want to prove an hypothesis
- Assume the opposite of the hypothesis and
- End up with a contradiction
- Conclude that the hypothesis is true

As an example, we will show that $\sqrt{2}$ is irrational:
$\sqrt{2} \in \mathbb{Q} \to \sqrt{2} = \frac{a}{b}$ irreducible fraction (that is in its lowest terms).
$\sqrt{2} = \frac{a}{b} \to 2 = \frac{a^2}{b^2} \to a^2 = 2b^2 \to a^2$ is even $\to a$ even $\to a = 2r \to$
$a^2 = 4r = 2b^2 \to b^2$ even $\to b$ even.

If a and b are even, then $\frac{a}{b}$ is not an irreducible fraction $\to$ contradiction! $\sqrt{2} \notin \mathbb{Q}$.

INEQUALITIES, ABSOLUTE VALUE

Order properties on $\mathbb{R}$
$a, b, c \in \mathbb{R}$

1. Only and only one of the following statements holds: $a < b,\ a = b,\ a > b$
2. If $a < b$ and $b < c$, then $a < c$
3. If $a < b$ then $a + c < b + c$ for any real number c
4. If $a < b$ and $c > 0$ then $ac < bc$
   If $a < b$ and $c < 0$ then $ac > bc$

Property. Among two different real numbers there exist infinite rational numbers and infinite irrational numbers.

Definition 1.1.1 *The* **Absolute Value** *of $x \in \mathbb{R}$ is:*

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

We have an alternative definition: $|x| = \sqrt{x^2}$.

The geometrical idea of the absolute value is that of distance. For instance, which points are at a distance 3 from 0? The answer is $|x| = 3$, that is $x = \pm 3$.

Definition 1.1.2 *If $x, y \in \mathbb{R}$ the* **distance** *between x and y is*

$$d(x, y) = |x - y|.$$

Properties of absolute value
$x, y \in \mathbb{R}$

1. $|x| = 0 \iff x = 0$
2. $|-x| = |x|$
3. $|xy| = |x||y|$
4. $|x + y| \leq |x| + |y|$
5. $||x| - |y|| \leq |x - y|$

Intervals on $\mathbb{R}$

Open interval: (a, b) means all x such that $a < x < b$.

Closed interval: $[a, b]$ means all x such that $a \leq x \leq b$.

Half-open interval: $[a, b)$ means all x such that $a \leq x < b$.

Half-open interval: $(a, b]$ means all x such that $a < x \leq b$.

Half-open interval: $[a, \infty)$ means all x such that $a \leq x < \infty$.

Open interval: (a, ∞) means all x such that $a < x$.

Half-open interval: $(-\infty, b]$ means all x such that $x \leq b$.

Open interval: $(-\infty, b)$ means all x such that $x < b$.

Open interval: $(-\infty, \infty)$ means all x such that $x \in \mathbb{R}$.

Note. ∞ is not a real number, it indicates that the interval extends without limit.

Definition 1.1.3 *Let A be a set of numbers*

- *The* **supremum** *of A is the least element of $\mathbb{R}$ that is greater than or equal to all elements of A. Also called the least upper bound (LUB).*
- *The* **infimum** *of A is the greatest element of $\mathbb{R}$ that is less than or equal to all elements of A. Also called the greatest lower bound (GLB).*
- *If the supremum belongs to A it is also the* **maximum***, (greatest element) of A.*
- *If the infimum belongs to A it is also the* **minimum***, (smallest element) of A.*

1.2 Elementary functions

The idea of a function f on the reals is a rule that associates to each number x a unique real number $f(x)$:

$$x \to y = f(x)$$

x: independent variable, argument or input.
y: dependent variable, value of the function or output.

Definition 1.2.1 *Let $D \subset \mathbb{R}$ be a set. A* **function** *f with* **domain** *D is a rule which assigns a unique real number $f(x)$ to each number x in D.*
D is called the **domain** *of f: $\mathrm{Dom}(f) = D$.*
The set of all outputs of f is called the **range** *or the* **image** *of f:*
$\mathrm{Im}(f) = \{y \in \mathbb{R} : \exists x \in D, f(x) = y\}$.

Notation: $f: D \subset \mathbb{R} \to \mathbb{R}$.

To visualize a function we can sketch its graph:

Definition 1.2.2 *Let f be a function with domain D, the set of all points $(x, f(x)) \in \mathbb{R}^2$ with $x \in D$ is called the* **graph** *of f.*

How can we recognize a graph of a function? we can plot vertical lines, then a curve which intersects each vertical line at most once is the graph of a function.

For instance, $y^2 = x$ has two solutions $y = \pm\sqrt{x}$, there are two intersections with vertical lines for $x > 0$, therefore is not the graph of a function.

If we plot vertical lines, we can see that the domain of a function is the set of x_0 such that the vertical line $x = x_0$ meets the graph. In the same way, plotting horizontal lines, we can visualize the image as the set y_0 such that the horizontal line $y = y_0$ meets the graph.

A function may be defined by a formula, a graph or a description.

Definition 1.2.3 *Let $f: D \to \mathbb{R}$ be a function:*

1. *f is an* **injective** *function if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$*
   or equivalently if $f(x_1) = f(x_2)$ then $x_1 = x_2$,
   that is, if it associates distinct arguments with distinct values.
2. *$f: D \to \mathbb{R}$ is called* **surjective or onto** *if $\mathrm{Im}(f) = \mathbb{R}$.*
   $f: D \to D$ is called **surjective or onto** *if $\mathrm{Im}(f) = D$.*
3. *f is a* **bijective** *function or* **one-to-one** *if it is injective and surjective at the same time.*

1.3 Limits

If $f(x)$ is a function defined for all x near x_0, but not necessarily at $x = x_0$ itself, if the value $f(x)$ of f approximates a number L as x gets closer to a number x_0, we say that L **is the limit of** $f(x)$ **as** x **approaches** x_0.

Let us see the behaviour of the following function near $x = 5$:

$$f(x) = \frac{2x^2 - 7x + 3}{x - 3}$$

x	2.9	2.9999	2.999999	$\to 3 \leftarrow$	3.000001	3.0001	3.1
$f(x)$	4.8	4.9998	4.999998	$\to 5 \leftarrow$	5.000002	5.0002	5.2

$$\lim_{x \to 3} f(x) = 5$$

$$f(x) = \frac{2x^2 - 7x + 3}{x - 3} = \frac{(2x-1)(x-3)}{x-3} \underset{x \neq 3}{=} 2x - 1.$$

Definition 1.3.1 (Weierstrass, $\epsilon - \delta$ definition) *Let f be a function defined on an open interval containing x_0 (except possibly at x_0) and let L be a real number, then*

$$\lim_{x \to x_0} f(x) = L,$$

if $\forall \epsilon > 0,\ \exists \delta > 0,\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$

Note. We can also use the $\varepsilon - \delta$ to prove that $\lim_{x \to 3}(2x - 1) = 5$.

Note. A useful general rule is to write down $f(x) = L$ and then to express it in terms of $x - x_0$ as much as possible, by writing $x = (x - x_0) + x_0$.

Note. In the $\varepsilon - \delta$ def, L and x_0 are finite numbers. We have similar definitions for $x \to \pm\infty$ and also if the limit is not a finite number. We define also **one-sided limits:** from the right of x_0: $\lim_{x \to x_0^+} f(x)$, from the left of x_0: $\lim_{x \to x_0^-} f(x)$.

Indeterminate forms

$$\infty - \infty,\ \frac{\infty}{\infty},\ \infty \cdot 0,\ \frac{0}{0},\ \infty^0,\ 1^\infty,\ 0^0.$$

Unless we arrive at an indetermination, we have the following properties.

Properties

Assume that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist, where x_0 is a finite number or $\pm\infty$

1. $\lim_{x\to x_0}(c_1 f(x) + c_2 g(x)) = c_1 \lim_{x\to x_0} f(x) + c_2 \lim_{x\to x_0} g(x), \quad c_1, c_2 \in \mathbb{R}$
2. $\lim_{x\to x_0}(f(x)g(x)) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x)$
3. $\lim_{x\to x_0}\left(\frac{1}{f(x)}\right) = \frac{1}{\lim_{x\to x_0} f(x)}$, if $\lim_{x\to x_0} f(x) \neq 0$
4. Replacement rule: if f and g agree for all x near x_0 (not necessarily including x_0), then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$
5. Composite function rule: if $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to L} h(x) = h(L)$, then $\lim_{x\to x_0} h(f(x)) = h(\lim_{x\to x_0} f(x))$

Basic limits. Let x_0 be a finite number

a) $\lim_{x\to x_0} c = c, \qquad \lim_{x\to\pm\infty} c = c$

b) $\lim_{x\to x_0} x = x_0, \qquad \lim_{x\to\pm\infty} x = \pm\infty, \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0$

c) $\lim_{x\to x_0} x^n = x_0^n$, $n \in \mathbb{N}$

d) $\lim_{x\to x_0} \sqrt[n]{x} = \sqrt[n]{x_0}$, ($\forall x_0$ in its domain)

e) $\lim_{x\to x_0} f(x) = f(x_0) \to$ trigonometric functions on their domain

Lemma 1.3.2 (Squeeze or Sandwich Lemma) *Let I be an interval such that $x_0 \in I$. Let f, g and h be functions defined on I, except possibly at x_0 itself. If $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L$ and, $\forall x \in I$, $x \neq x_0$, $h(x) \leq f(x) \leq g(x)$. Then*

$$\lim_{x\to x_0} f(x) = L.$$

Note. We can use the Sandwich Lemma to prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

1.4 Continuity

A continuous function is a function for which intuitively, small changes in the input result in small changes in the output

Definition 1.4.1 *Let f be a function defined on $(x_0 - p, x_0 + p)$, $p > 0$*

$$f \textit{ is } \textbf{\textit{continuous}} \textit{ at } x_0 \iff \lim_{x \to x_0} f(x) = f(x_0).$$

Notice that the function must be defined at x_0 to be continuous at that point. If the function is not continuous at x_0 we say that it is **discontinuous** at that point:

- $f(x)$ is discontinuous at x_0 if $\begin{cases} \lim_{x \to x_0} f(x) \text{ does not exist, or} \\ \text{the limit exist but is not equal to } f(x_0) \end{cases}$

Definition 1.4.2 *A function $f(x)$ is continuous on*

- $\mathbb{R}$*, if it is continuous at every point*
- (a, b)*, if it is continuous at each point of the interval*
- $[a, b]$*, if it is continuous on (a, b) and*

$$f(a) = \lim_{x \to a^+} f(x), \;\; f(b) = \lim_{x \to b^-} f(x)$$

Basic properties
Let f, g be continuous at x_0, then the following functions are continuous at x_0

1. $c_1 f + c_2 g, \quad c_1, c_2 \in \mathbb{R}$
2. fg
3. $\dfrac{1}{f}$, if $f(x_0) \neq 0$
4. Composite function: if g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0

Some continuous functions
The following functions are continuous on their domains

a) $p(x)$ polynomials, $\dfrac{p(x)}{q(x)}$ rational functions, $\sqrt[n]{x}$

b) trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\arcsin(x), \dots$

c) hyperbolic functions: $\sinh(x)$, $\cosh(x), \dots$

d) $\exp(x)$ and $\ln(x)$

Theorem 1.4.3 (Intermediate Value Theorem) *If $f(x)$ is a continuous function on $[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = K$.*

Theorem 1.4.4 (Bolzano's Theorem) *If $f(x)$ is a continuous function on $[a, b]$ and $f(a) \cdot f(b) < 0$, then there is a $c \in (a, b)$ such that $f(c) = 0$.*

Definition 1.4.5 *A function f is* ***bounded*** *if the set of its values is bounded. That is, if there exists a number $M > 0$ such that $|f(x)| \leq M$, for all x in its domain.*

Theorem 1.4.6 (Extreme Value Theorem) *If $f(x)$ is a continuous function on the closed interval $[a, b]$, then f attains its maximum and minimum value. That is, there exist numbers x_m and x_M in $[a, b]$ such that:*

$$f(x_M) \geq f(x) \geq f(x_m) \quad \textit{for all } x \in [a, b].$$