Unit 3

Sequences and Series

3.1 Real Sequences

Definition 3.1.1 *A* **real sequence** $\{a_n\}$ *is a map that assigns to every* $n \in \mathbb{N}$ *a real number:*

$$a_n : \mathbb{N} \to \mathbb{R}$$

$a_1, a_2, a_3 \ldots$ are the terms of the sequence.
a_n is the general term.
The sequence can also begin with $n = 0$: $a_0, a_1, a_2 \ldots$.

Definition 3.1.2 *A sequence* $\{a_n\}$ *is* **convergent** *if* $\lim_{n\to\infty} a_n = L$, *for L finite.*

The **limit** *of a sequence* $\{a_n\}$ *is L if for every* $\epsilon > 0$ $\exists N \in \mathbb{N}$ *such that if* $n > N \Rightarrow |a_n - L| < \varepsilon$. *(There is an alternative def. for L infinite.)*

If the sequence is not convergent, we say that it is divergent.
The properties of limits of sequences are the same as the properties of limits of functions.

To **compute the limit** of a sequence we can use some techniques:

- Use the concept of the limit of a function:

  Let $f(x)$ be a function and $\{a_n\}$ the sequence $f(n) = a_n$.

  If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L1$

  We have all the tools of computing the limit of a function, such as the L'Hopital Rule.

- The **Sandwich lemma** of sequences:

  If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ (finite or infinite) and $\{c_n\}$ verifies $a_n \leq c_n \leq b_n, \quad \forall n \in \mathbb{N}$,

  then $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Definition 3.1.3 *A sequence $\{a_n\}$ is called:*

1. *bounded from above if $\exists\, C \in \mathbb{R}$ such that $a_n \leq C$.*
2. *bounded from below if $\exists\, C \in \mathbb{R}$ such that $a_n \geq C$.*
3. **bounded** *if it is bounded from above and below ($\exists\, C_1, C_2 \in \mathbb{R}$, s. t. $C_1 \leq a_n \leq C_2$).*

Definition 3.1.4 *A sequence $\{a_n\}$ is called:*

1. *monotonically increasing if $a_n < a_{n+1}$ (non decreasing if $a_n \leq a_{n+1}$).*
2. *monotonically decreasing if $a_n > a_{n+1}$ (non increasing if $a_n \geq a_{n+1}$).*
3. **monotonic** *if it is one of the previous cases.*

Theorem 3.1.5

$$\{a_n\} \text{ monotonic and bounded} \Rightarrow \{a_n\} \text{ convergent}$$

Theorem 3.1.6 (Stolz Test) *If the sequences $\{a_n\}$ and $\{b_n\}$ verify one of the following:*

1. *$\{b_n\}$ is monotonically increasing with $\lim\limits_{n\to\infty} b_n = \infty$,*
2. *$\{b_n\}$ is monotonically decreasing, with $b_n \neq 0$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.*

Whenever $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$, exists for L finite or infinite, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

Theorem 3.1.7 (Stirling's Formula)

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$$

3.2 Real Series

A series is the sum of a sequence of terms

For instance → the geometrical sum: $\sum_{n=0}^{N} r^n = \frac{r^{N+1}-1}{r-1}$.

Definition 3.2.1 *Let* $\{a_n\}$ *be a sequence, an (infinite)* **series** *is the sum of all its terms:*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots .$$

The **partial sum** *of* n *terms is* $S_n = a_1 + a_2 + a_3 + \cdots a_n$.
If the sequence $\{S_n\}$ *of partial sum converges to the limit* S*, then we say that the series* $\sum_{n=1}^{\infty} a_n$ **converges**, *and* S *is called the sum of the series:*

$$S = \lim_{n\to\infty} S_n = a_1 + a_2 + a_3 + a_4 + \cdots .$$

Otherwise, we say that the series **diverges.**

Properties

1. $\sum a_n$ and $\sum b_n$ conv $\Rightarrow \sum (c_1 a_n + c_2 b_n) = c_1 \sum a_n + c_2 \sum b_n$ conv.
2. $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ div.
3. $\sum a_n$ conv $\Rightarrow \lim_{n\to\infty} a_n = 0$. (But $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum a_n$ conv)

Theorem 3.2.2 *The* **geometrical sum** *converges if* $0 < |r| < 1$*, in this case*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Theorem 3.2.3 *The* **telescoping series** $(a_n = b_n - b_{n+1})$

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots$$

verifies $S_n = b_1 - b_{n+1}$.
This series converges $\iff \lim_{n\to\infty} b_n < \infty$ *and*

$$S = b_1 - \lim_{n\to\infty} b_n.$$

Theorem 3.2.4 *The* **p-series** *(*$p = 1$ *is the harmonic series)*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots .$$

1. *converges if* $p > 1$.
2. *diverges if* $0 < p \leq 1$.

CONVERGENCE TEST FOR SERIES

1. **Direct comparison test:** $\{a_n\}$ and $\{b_n\}$ two sequences of positive terms

$$0 < a_n \le b_n,\ \forall\, n \quad \longrightarrow \quad \begin{array}{c} \sum b_n \text{ conv} \Rightarrow \sum a_n \text{ conv} \\ \sum a_n \text{ div} \Rightarrow \sum b_n \text{ div} \end{array}$$

2. **Limit comparison test:** $\{a_n\}$ and $\{b_n\}$ two sequences of positive terms

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \quad L \text{ finite and positive}$$
$$\Downarrow$$
$$\sum a_n \text{ and } \sum b_n \text{ have the same behaviour}$$
$$\text{both converge or both diverge}$$

3. **Root test:** $\{a_n\}$ sequence of positive terms

$$\lim_{n\to\infty} \sqrt[n]{a_n} < 1 \Rightarrow \sum a_n \text{ conv}$$
$$\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \Rightarrow \sum a_n \text{ div}$$
$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \text{ the test does not conclude}$$

4. **Quotient test:** $\{a_n\}$ sequence of positive terms

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n \text{ conv}$$
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \sum a_n \text{ div}$$
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \text{ the test does not conclude}$$

5. **Leibniz test for alternating series:** $\{a_n\}$ sequence of positive terms

$$\text{If } a_{n+1} \le a_n \text{ and } \lim_{n\to\infty} a_n = 0$$
$$\Downarrow$$
$$\text{The alternating series } \sum (-1)^n a_n \text{ converges conditionally}$$
$$\left(\sum (-1)^{n+1} a_n\right)$$

Definition 3.2.5

AC. $\sum a_n$ *is* **absolutely convergent** *if* $\sum |a_n|$ *is convergent.*

CC. *If* $\sum a_n$ *is convergent but* $\sum |a_n|$ *is divergent then the series is* **conditionally convergent**.

Absolute convergence	$\Longrightarrow$	Conditional convergence
No conditional convergence	$\Longrightarrow$	No absolute convergence

ERROR:

when we approximate the sum of an alternating series by its first n-terms, then

$$S = S_N + R_N = \sum_{n=1}^{N} (-1^n) a_n + R_N \quad \Rightarrow \ |R_N| \leq a_{N+1}$$

Note. We can differentiate or integrate an infinite series to obtain another series.

3.3 Power Series

Definition 3.3.1 *A* **power series** *at* x_0 *is an infinite series of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

Theorem 3.3.2 (Convergence of a power series)
A power series at x_0 *verifies only one of the following:*

1. *The series converges only at* x_0
2. *There is a real number* $\rho > 0$ *such that the series is*
   - *absolutely convergent for* $|x - c| < \rho$
   - *divergent for* $|x - c| > \rho$
3. *The series is absolutely convergent for every* $x \in \mathbb{R}$

Note. ρ is the **radius of convergence** of the power series. $\rho = 0$, $\rho < \infty$ or $\rho = \infty$. The set of all x for which the series converges is the **interval of convergence** of the series. The radius can be computed by the following formulas:

- $\dfrac{1}{\rho} = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$
- $\dfrac{1}{\rho} = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$, if this limit exists

Theorem 3.3.3 *If the power series* $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *has a radius of convergence* $\rho > 0$*, then* $f(x)$ *is continuous, differentiable and integrable on* $(x_0 - \rho, x_0 + \rho)$*. The derivative and the integral are computed term by term. Both of them have the same radius of convergence as* f *does. The interval of convergence may be different, because of the end points* $(x = x_0 \pm \rho)$*.*

Properties.
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$
2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{Nn}$
3. $c_1 f(x) + c_2 g(x) = \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n) x^n$

Definition 3.3.4 *If* f *has all the derivatives at* x_0 *then the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is called the **Taylor series** *of* f *at* x_0 *(for* $x_0 = 0$ *also called the Mac Laurin series of* f*).*

Theorem 3.3.5 *If* f *has all the derivatives on an open interval* I *containing* x_0 *then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

if and only if there exists ξ *between* x *and* x_0 *such that*

$$\lim_{n \to \infty} R_n(x) = \lim_{n \to \infty} \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} = 0, \quad \forall x \in I.$$

TAYLOR SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \cdots, \quad -\infty < x < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdots, \quad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \cdots, \quad -\infty < x < \infty$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)} \cdots, \quad -1 \le x \le 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n \cdots, \quad -1 < x < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} \cdots, \quad -1 < x \le 1$$