Unit 4

Integration

4.1 Antiderivatives

THE DEFINITE INTEGRAL

If $f(x)$ is a continuous and nonnegative function on the interval $x \in [a, b]$, then the **definite integral**

$$\int_a^b f(x)$$

represents the **area under the graph** of the function $f(x)$, and over the x-axis on the interval $x \in [a, b]$.

For any function $f(x)$, the definite integral represents the **sum of the signed areas** between the function and the x axis. The idea to compute the definite integral is to divide the interval into n subintervals and approximate the function by a constant (the lowest, the greatest, the midpoint or any value of the function on the interval). Then, we compute the area of n rectangles, that is very easy. If we approximate $f \simeq f(x_i)$ at the i-th interval, with all the intervals with the same length, Δx, then we can approximate the integral as

$$\int_a^b f(x) \simeq f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

If we approximate the function on the interval by the following values of the function, we obtain what is called, for that partition of the interval

The lowest value on each subinterval	$\rightarrow$ Lower sum of f
The greatest value on each subinterval	$\rightarrow$ Upper sum of f
Any value value on each subinterval	$\rightarrow$ Riemann sum of f

When we take the limit of $\Delta x \rightarrow 0$ if all the sums coincide, we say that the function is **Riemann integrable**.
Note. Any piecewise-continuous function is integrable.

We have the following definition of the definite integral:

Definition 4.1.1 *Given an integrable function $f(x)$ on the interval $[a,b]$, divide the interval into n subintervals of equal length Δx, choose any point $x_i^\star$ in each subinterval, then we define the definite integral or Riemann integral of $f(x)$ from a to b as*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x.$$

Note. The numerical methods are based on this concept of computing the integral by approximating the value of the function on each subinterval by a constant or by a polynomial, that is easy to integrate.

Properties of the integral

1. $\int_a^b c_1 f + c_2 g = c_1 \int_a^b f + c_2 \int_a^b g$
2. $\int_a^b f = \int_a^c f + \int_c^b f$
3. $\int_a^b f = -\int_b^a f$
4. $\int_a^a f = 0$
5. $\int_a^b fg \neq \int_a^b f \int_a^b g$
6. $f \geq g \Rightarrow \int_a^b f \geq \int_a^b g$
7. $f \geq 0 \Rightarrow \int_a^b f \geq 0$
   if $f \leq 0 \Rightarrow \int_a^b f \leq 0$
8. $\left| \int_a^b f \right| \leq \int_a^b |f|$
9. $m \leq f(x) \leq M,\ \forall x \in [a,b] \Rightarrow$
   $m(b-a) \leq \int_a^b f(x) \leq M(b-a)$

THE INDEFINITE INTEGRAL

Geometrically the problem of differentiation arises when we want to find the slope of a curve and the problem of integration when computing the area under a curve but Newton found out that **differentiation and integration are inverse processes:**

- **Differentiation:** Given a function $F(x)$, find a function $f(x)$ satisfying

$$\frac{dF(x)}{dx} = f(x).$$

- **Integration:** Given a function $f(x)$, find a function $F(x)$ satisfying

$$\frac{dF(x)}{dx} = f(x).$$

A function $F(x)$ solving the second problem is called an **antiderivative primitive** or an **indefinite integral** of $f(x)$.
The problem of differentiation has always solution but the problem of integration does not always have a solution and, in general, is more complicated.

INTEGRATION TECHNIQUES

We will see the most usual techniques to compute the integral (definite or indefinite) of a function.

Basic antiderivatives

$$\int x^n = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int e^{ax} = \frac{1}{a} e^{ax} + c$$

$$\int \sin x = -\cos x + c$$

$$\int \cos x = \sin x + c$$

$$\int \frac{1}{\cos^2 x} = \tan x + c$$

$$\int \frac{1}{\sin^2 x} = -\cot x + c$$

$$\int \frac{1}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \sinh x = \cosh x + c$$

$$\int \cosh x = \sinh x + c$$

Integration by change of variables or by substitution (CV)

- Definite integral

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt$$

- Indefinite integral

$$\int f(x)dx = \int f(g(t))g'(t)dt$$

at the end, undo the change

Integration by parts (IBP)

- Definite integral

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

- Indefinite integral

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\int udv = uv - \int vdu$$

Integration of rational functions: partial fraction decomposition

$$\int \frac{P(x)}{Q(x)}\,dx \to P,\ Q \text{ polynomials.}$$

- If degree of $P \geq$ degree of $Q \Rightarrow$ we must **divide** the polynomials:
  $P(x) = Q(x)C(x) + R(x) \to$
  $$\int \frac{P(x)}{Q(x)}dx = \int C(x) + \int \frac{R(x)}{Q(x)}dx.$$
- $\displaystyle\int \frac{R(x)}{Q(x)}dx$ with $deg(R(x)) < deg(Q(x))$:

  i) First, we must check that the integral is not immediate, that is, if is one of the following:

  ln type $\displaystyle \to \int \frac{2x+3}{x^2+3x+8}dx = \ln|x^2+3x+8| + c.$

  arctan type $\displaystyle \to \int \frac{dx}{x^2+8} = \frac{1}{\sqrt{8}} \arctan \frac{x}{\sqrt{8}} + c.$

  ii) if not $\to$ **Do partial fraction decomposition:**

Factor in denominator	Term in partial fraction decomposition
$x-b$	$\dfrac{A}{x-b}$
$(x-b)^k$	$\dfrac{A_1}{x-b} + \dfrac{A_2}{(x-b)^2} + \cdots \dfrac{A_k}{(x-b)^k}, \quad k = 1, 2, 3, \cdots$
$(x-a)^2+b^2$	$\dfrac{Ax+B}{(x-a)^2+b^2}$
$\left((x-a)^2+b^2\right)^k$	$\dfrac{A_1x+B_1}{(x-a)^2+b^2} + \dfrac{A_2x+B_2}{\left((x-a)^2+b^2\right)^2} + \cdots + \dfrac{A_kx+B_k}{\left((x-a)^2+b^2\right)^k}, \quad k = 1, 2, 3, \cdots$

For each factor in the denominator add the corresponding term of the table and compute the unknowns $(A, B, A_1, B_1, A_2, B_2, \cdots)$ by setting equal denominators. After, compute the integrals of each term.

From now on $R = \dfrac{P}{Q}$ means a rational function of its variables, $P,\ Q$ are polynomials.

Irrational functions or integrals involving roots

Do a change of variables that eliminates the roots.

$$\int R\left[\left(\frac{ax+b}{cx+d}\right)^{p_1/q_1}, \cdots, \left(\frac{ax+b}{cx+d}\right)^{p_r/q_r}\right] \to t^m = \frac{ax+b}{cx+d},\ m = lcm(q_1, \cdots, q_r).$$

$lcm \to$ least common multiple.

Integrals involving trigonometric functions

$\int \sin^{2n} x,\ \int \cos^{2n} x \to$ double angle formulas: $\cos 2x = \cos^2 x - \sin^2 x$

$\int \sin^{2n+1} x = \int \sin^{2n} x \sin x = \int (1-\cos^2 x)^n \sin x$

$\int \cos^{2n+1} x = \int \cos^{2n} x \cos x = \int (1-\sin^2 x)^n \cos x$

$\int \sin mx \cos nx \to$ trig formulas

$$\int R(\sin x, \cos x) \to \begin{cases} R \text{ odd in } \sin x \to & t = \cos x \\ R \text{ odd in } \cos x \to & t = \sin x \\ R \text{ even in } \cos x \text{ and } \sin x \to & t = \tan x \\ \text{Rest of problems} \to & t = \tan x/2, \\ \left(\sin x = \frac{2t}{1+t^2},\ \ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2}{1+t^2}dt\right) & \end{cases}$$

Some change of variables

1. $\int R(x, \sqrt{x^2+a^2}) \to x = a \tan t$
2. $\int R(x, \sqrt{x^2-a^2}) \to x = \dfrac{a}{\cos t}$
3. $\int R(x, \sqrt{a^2-x^2}) \to x = a \sin t$

4.2 The Fundamental Theorem of Calculus

Let f be integrable on [a,b], $F(x) = \int_a^x f(t)dt$ is an antiderivative of $f(x)$ defined on $[a, b]$.

Theorem 4.2.1

$$f \text{ integrable on } [a, b] \Rightarrow F \text{ continuous on } [a, b]$$

Theorem 4.2.2 (The Fundamental Theorem of Calculus, FTC)
Let f be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, defined $\forall x \in [a, b]$.
If f is continuous at $c \in [a, b] \Rightarrow F$ is differentiable at c and $F'(c) = f(c)$.
If f is continuous $\forall x \in [a, b] \Rightarrow F$ is differentiable $\forall x \in [a, b]$ and $F'(x) = f(x)$.

Theorem 4.2.3 (Barrow's Rule) *Let f and g be continuous on $[a, b]$ and g differentiable on (a, b), such that $g'(x) = f(x)$, $\forall x \in (a, b)$, then*

$$\int_a^b f = \int_a^b g' = g(b) - g(a).$$

Theorem 4.2.4 (FTC generalized) *Let $F(x) = \int_a^x f$, with f integrable*

- *Let $H(x) = F(g(x)) = \int_a^{g(x)} f$, then if g is differentiable, we have*

$$H'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

- *Let $H(x) = \int_{l(x)}^{g(x)} f$, then if g and l are differentiable, we have*

$$H'(x) = f(g(x))g'(x) - f(l(x))l'(x).$$

4.3 Applications of the Integral

AREAS

- Area between the **graph of a function**, the x-axis, between a and b:

$$A = \int_a^b |f| dx$$

- Area between the **graphs of two functions** f, g, between a and b:

$$A = \int_a^b |f - g| dx$$

- Area using **parametric equations:** the area between the graph of $x = x(t)$, $y = y(t)$ and the x-axis between $t = t_0$ and $t = t_1$ is:

$$A = \left| \int_{t_0}^{t_1} y(t) x'(t) dt \right|$$

- Area using **polar coordinates:** the area of the graph of $r = r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_\alpha^\beta \frac{1}{2} r^2(\theta) d\theta$$

VOLUMES

- **Volume by parallel cross-sections:** if $A(x)$ is the area of parallel cross-sections over the entire length of a solid, the volume between $x = a$ and $x = b$ is

$$V = \int_a^b A(x) dx$$

- **The Disk method:** the volume of a solid of revolution obtained by rotating $|f(x)|$ about the x-axis between $x = a$ and $x = b$ is

$$V = \int_a^b \pi (f(x))^2 dx$$

- **The Shell method:** the volume of a solid of revolution obtained by rotating $f(x) \geq 0$, $x \in [a, b]$, $a \geq 0$, about the y-axis is

$$V = 2\pi \int_a^b x f(x) dx$$

LENGTHS

- The **length of an arc of a curve** $f(x)$ between $x = a$ and $x = b$ is

$$L(f) = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

- If the curve is given in **parametric form**, the length is

$$L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

IMPROPER INTEGRAL

Definition 4.3.1 *The following integral*

$$\int_a^\infty f(x) = \lim_{N\to\infty} \int_a^N f(x),$$

is called an **improper integral** *of f. If the limit is finite we say that the integral* **converges**
otherwise we say that the integral diverges.

Theorem 4.3.2 (Integral test for series) *Consider $f \geq 0$ a monotone decreasing function defined for $x \geq 1$. Let $a_n = f(n)$, then*

$$\sum_{n=1}^\infty a_n \quad \text{and} \quad \int_1^\infty f(x)\, dx,$$

have the same behaviour, or both converge or both diverge.