UNIVERSIDAD CARLOS III DE MADRID

Escuela Politécnica Superior

Departamento de Matemáticas

PROBLEMS, CALCULUS I, 1st COURSE

2. DIFFERENTIAL CALCULUS IN ONE VARIABLE

BACHELOR IN:
Audiovisual System Engineering
Communication System Engineering
Telematics Engineering

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2 Differential calculus in one variable

2.1 Derivatives

Problem 2.1.1 Let f, g be differentiable functions on all $I\!R$. Write down the derivative of the following functions on their domain:

$$i)\quad h(x) = \sqrt{f^2(x) + g^2(x)}, \qquad ii)\quad h(x) = \operatorname{arctg}\left(\frac{f(x)}{g(x)}\right),$$

$$iii)\quad h(x) = f(g(x))\mathrm{e}^{f(x)}, \qquad iv)\quad h(x) = \log(g(x)\sin(f(x))),$$

$$v)\quad h(x) = (f(x))^{g(x)}, \qquad vi)\quad h(x) = \frac{1}{\log(f(x) + g^2(x))}.$$

Problem 2.1.2

i) Build a continuous function for all $I\!R$ that vanishes for $|x| \geq 2$ and takes value one for $|x| \leq 1$.

ii) Build another one which is also differentiable.

Problem 2.1.3 Starting from the hyperbolic functions $\sinh x$ and $\cosh x$, we define $\operatorname{tgh} x = \dfrac{\sinh x}{\cosh x}$ and $\operatorname{sech} x = \dfrac{1}{\cosh x}$. Prove the formulas

$$i)\quad (\sinh x)' = \cosh x, \qquad ii)\quad (\cosh x)' = \sinh x,$$

$$iii)\quad (\operatorname{tgh} x)' = \operatorname{sech}^2 x, \qquad iv)\quad (\operatorname{sech} x)' = -\operatorname{sech} x \operatorname{tgh} x.$$

Problem 2.1.4 Check that the following functions fulfill the specified differential equations, where c, c_1 and c_2 are constants.

$$i)\quad f(x) = \frac{c}{x}, \qquad xf' + f = 0;$$

$$ii)\quad f(x) = x \operatorname{tg} x, \qquad xf' - f - f^2 = x^2;$$

$$iii)\quad f(x) = c_1 \sin 3x + c_2 \cos 3x, \qquad f'' + 9f = 0;$$

$$iv)\quad f(x) = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-3x}, \qquad f'' - 9f = 0;$$

$$v)\quad f(x) = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{5x}, \qquad f'' - 7f' + 10f = 0;$$

$$vi)\quad f(x) = \log(c_1 \mathrm{e}^{x} + \mathrm{e}^{-x}) + c_2, \qquad f'' + (f')^2 = 1.$$

Problem 2.1.5 Prove the identities

$$i)\quad \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}, \qquad x > 0;$$

$$ii)\quad \operatorname{arctg}\frac{1+x}{1-x} - \operatorname{arctg} x = \frac{\pi}{4}, \qquad x < 1;$$

$$iii)\quad 2\operatorname{arctg} x + \arcsin\frac{2x}{1+x^2} = \pi, \qquad x \geq 1.$$

Hint: differentiate and substitute at some point of the interval. The result is *not* true outside the specified intervals.

Problem 2.1.6 Find the value of $a \in I\!R$ for which the parabola $f(x) = ax^2$ is tangent to the graph of $g(x) = \log x$, and write the equation of the common tangent.

Problem 2.1.7 Find the points at which the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent.

Problem 2.1.8 Find the angle spanned by the left and right tangents at the origin to the plot of the function

$$f(x) = \begin{cases} \dfrac{x}{1 + e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Problem 2.1.9 Given the function

$$f(x) = \begin{cases} (3 - x^2)/2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1, \end{cases}$$

i) study its continuity and differentiability;

ii) may we apply the Mean Value Theorem on [0,2]? If the answer is positive, find the point (or points) from the thesis of the theorem.

Problem 2.1.10 Study the continuity and differentiability of the function

$$f(x) = \sqrt{x+2}\,\arccos(x+2).$$

Problem 2.1.11 Find the minimum value of a for which the function $f(x) = |\alpha x^2 - x + 3|$ is differentiable on all $I\!R$.

Problem 2.1.12 The function $f(x) = 1 - x^{2/3}$ vanishes at -1 and 1 and, nonetheless, $f'(x) \neq 0$ on $(-1, 1)$. Explain this apparent contradiction with the Rolle's Theorem.

Problem 2.1.13 Given the function $f(x) = \begin{cases} a + bx^2 & \text{if } |x| \leq c \\ |x|^{-1} & \text{if } |x| > c \end{cases}$, with $c > 0$, find a and b so it is continuous and differentiable on the whole of $I\!R$.

Problem 2.1.14 Using the Mean Value Theorem, approximate $26^{2/3}$ and $\log(3/2)$.

Problem 2.1.15

i) If f is a differentiable function, find

$$\lim_{h \to 0} \frac{f(a + ph) - f(a - qh)}{h}.$$

ii) Can the previous limit exist for a function which is not differentiable at $x = a$?

iii) If f is an even and differentiable function, find $f'(0)$.

iv) If f is a twice differentiable function, find

$$\lim_{h \to 0} \frac{f(a + h) + f(a - h) - 2f(a)}{h^2}.$$

Problem 2.1.16 Find the limits of the problems 1.3.2 and 1.3.3 using L'Hôpital's Rule, writing them previously in the appropriate form.

Problem 2.1.17 Find the following limits:

i) $\lim_{x\to 0} \dfrac{e^x - \sin x - 1}{x^2}$, *ii)* $\lim_{x\to 0} \dfrac{\log|\sin 7x|}{\log|\sin x|}$,

iii) $\lim_{x\to 1} \log x \cdot \log(x-1)$, *iv)* $\lim_{x\to\infty} x^{1/x}$,

v) $\lim_{x\to 0} \dfrac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$, *vi)* $\lim_{x\to\infty} x\Big(\operatorname{tg}(2/x) - \operatorname{tg}(1/x)\Big)$.

Problem 2.1.18 Find the following limits:

i) $\lim_{x\to\infty} \dfrac{x^{x-1}}{(x-1)^x}$, *ii)* $\lim_{x\to 0} \dfrac{1+\sin x - e^x}{\operatorname{arctg} x}$,

iii) $\lim_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3}$, *iv)* $\lim_{x\to 0} (1+x^2)^{3/(2\arcsin x)}$,

v) $\lim_{x\to 1/2} (2x^2+3x-2)\operatorname{tg}(\pi x)$, *vi)* $\lim_{x\to 0} \dfrac{2x\sin x}{\sec x - 1}$,

vii) $\lim_{x\to -\infty} (\sqrt{x^2+1} - \sqrt{x^2-4x})$, *viii)* $\lim_{x\to 0^+} x^{1/\log x}$.

Problem 2.1.19 Let h be a twice differentiable function, and let

$$f(x) = \begin{cases} h(x)/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Assuming that f is continuous, find $h(0)$, $h'(0)$ and $h''(0)$.

Problem 2.1.20 Find a so that $\lim_{x\to 0} \dfrac{e^{ax} - e^x - x}{x^2}$ is finite and find the value of the limit.

Problem 2.1.21 Calculate the following limits:

i) $\lim_{x\to\infty} x\Big((1+1/x)^x - e\Big)$, *ii)* $\lim_{x\to\infty} \dfrac{(1+1/x)^{x^2}}{e^x}$,

iii) $\lim_{x\to\infty} \Big(\dfrac{2^{1/x}+18^{1/x}}{2}\Big)^x$, *iv)* $\lim_{x\to\infty} \Big(\dfrac{1}{p}\sum_{i=1}^{p} a_i^{1/x}\Big)^x$, $p \in I\!N$, $a_i > 0$.

Problem 2.1.22 Given a differentiable function f, which satisfies $\lim_{x\to 0} \dfrac{f(2x^3)}{5x^3} = 1$,

i) justify that $f(0) = 0$;

ii) prove that $f'(0) = 5/2$;

iii) calculate $\lim_{x\to 0} \dfrac{(f\circ f)(2x)}{f^{-1}(3x)}$.

Problem 2.1.23 Using the Mean Value Theorem compute the limit

$$\lim_{x\to\infty}\left[(1+x)^{1+\frac{1}{1+x}}-x^{1+\frac{1}{x}}\right].$$

Problem 2.1.24

i) Let $f(x)=\sin x$. Calculate the values of x for which we have $(f^{-1})'(x)=5/4$.

ii) Same question with $g(x)=\log(x+\sqrt{x^2+1})$ and $(g^{-1})'(x)=2$.

Problem 2.1.25 The equation

$$\begin{cases} e^{-f}f'=2+\operatorname{tg}x \\ f(0)=1, \end{cases}$$

defines a differentiable one-to-one function f on the interval $[-\pi/4,\pi/4]$. We define the function $g(x)=f^{-1}(x+1)$. Find the limit

$$\lim_{x\to 0}\frac{e^x-e^{-\sin x}}{g(x)}.$$

Problem 2.1.26

i) Let $f:[a,b]\longrightarrow \mathbb{R}$, differentiable. If f has $k\geq 2$ roots on $[a,b]$, then f' has at least $k-1$ roots on $[a,b]$.

ii) If f is n times differentiable on $[a,b]$ and vanishes at $n+1$ different points on $[a,b]$, prove that $f^{(n)}$ vanishes at least once on $[a,b]$.

Problem 2.1.27 Calculate how many different solutions do the following equations have on the given intervals:

i)	$x^7+4x=3$,	on $\mathbb{R}$;	*ii)*	$x^5=5x-6$,	on $\mathbb{R}$;
iii)	$x^4-4x^3=1$,	on $\mathbb{R}$;	*iv)*	$\sin x=2x-1$,	on $\mathbb{R}$;
v)	$x^x=2$,	on $[1,\infty)$;	*vi)*	$x^2=\log(1/x)$,	on $(1,\infty)$.

2.2 Extrema of functions

Problem 2.2.1 Let the function $f(x)=|x^3(x-4)|-1$.

i) Study its continuity and differentiability.

ii) Find its local extrema.

iii) Prove that the equation $f(x)=0$ has a single solution on the interval $[0,1]$.

Problem 2.2.2 A tomato sauce company wants to manufacture cylindrical cans of fixed volume V. What must be the relation between the radius r of the basis and its height so the minimum amount of material is employed?

Problem 2.2.3 Find the area of the rectangle, with sides parallel to the axes and inscribed in the ellipse $(x/a)^2+(y/b)^2=1$ with maximum area.

Problem 2.2.4 Find the area of the triangle formed by a tangent to the parabola $y=6-x^2$ and the positive semiaxes which has maximum area.

Problem 2.2.5 A right triangle ABC has the A vertex at the origin, B on the circumference $(x-1)^2+y^2=1$, and the side AC on the horizontal axis. Calculate C such that the area of the triangle is maximum.

Problem 2.2.6 Let $P = (x_0, y_0)$ be a point in the first quadrant. Draw a straight line which passes through P and cuts the axes at $A = (x_0+\alpha, 0)$ and $B = (0, y_0+\beta)$ respectively. Calculate $\alpha, \beta > 0$ such that the following magnitudes are minimized:

i) length of AB;

ii) length of OA plus OB;

iii) area of the triangle OAB.

Hint: $\beta = x_0 y_0/\alpha$.

Problem 2.2.7

i) Prove Bernoulli's inequality: $(1 + x)^a \geq 1 + ax$, for all $a \geq 1$, $x > -1$.

ii) Prove that $\mathrm{e}^x \geq 1 + x$ for all $x \in \mathbb{R}$.

iii) Prove that $\dfrac{x}{1+x} \leq \log(1 + x) \leq x$ for all $x > -1$.

Hint: minimize the appropriate functions.

Problem 2.2.8 Prove that if $a > 0$, the function $f(x) = \left(1 + \dfrac{a}{x}\right)^x$ increases, for $x > 0$, from 1 up to e^a.
Hint: use the previous estimates.

Problem 2.2.9 Prove by induction the inequality $n^n < n!\,\mathrm{e}^n$, for all $n \in \mathbb{N}$.

Problem 2.2.10

i) Prove that $\dfrac{\log x}{x} < \dfrac{1}{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$.

ii) Find as a conclusion that $\mathrm{e}^x > x^{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$.

Problem 2.2.11 Find the absolute maxima and minima of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ on the interval $[-2, 1]$.

2.3 Graphical representation

Problem 2.3.1 Prove that if f and g are convex, twice differentiable functions, and f is increasing, then $h = f \circ g$ is convex.

Problem 2.3.2

i) Sketch the plot of the function $f(x) = x + \log |x^2 - 1|$.

ii) From it, draw the plot of the functions

$$a)\quad g(x) = |x| + \log |x^2 - 1|, \qquad b)\quad h(x) = \Big|x + \log |x^2 - 1|\Big|.$$

Problem 2.3.3 Represent graphically the following functions:

i) $y = \mathrm{e}^x \sin x$, *ii)* $y = \sqrt{x^2 - 1} - 1$, *iii)* $y = x\mathrm{e}^{1/x}$,

iv) $y = x^2 \mathrm{e}^x$, *v)* $y = (x-2)x^{2/3}$, *vi)* $y = (x^2 - 1)\log \dfrac{1+x}{1-x}$,

vii) $y = \dfrac{x}{\log x}$, *viii)* $y = \dfrac{x^2-1}{x^2+1}$, *ix)* $y = \dfrac{\mathrm{e}^{1/x}}{1-x}$,

x) $y = \log[(x-1)(x-2)]$, *xi)* $y = \dfrac{\mathrm{e}^x}{x(x-1)}$, *xii)* $y = 2\sin x + \cos 2x$,

xiii) $y = \dfrac{x-2}{\sqrt{4x^2+1}}$, *xiv)* $y = \sqrt{|x-4|}$, *xv)* $y = \dfrac{1}{1+\mathrm{e}^x}$,

xvi) $y = \dfrac{\mathrm{e}^{2x}}{\mathrm{e}^x - 1}$, *xvii)* $y = \mathrm{e}^{-x}\sin x$, *xviii)* $y = x^2 \sin(1/x)$.

Problem 2.3.4 Represent graphically the following functions:

i) $f(x) = \min\{\log|x^3 - 3|, \log|x+3|\}$, *ii)* $g(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$,

iii) $h(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$, $a > 0$, *iv)* $k(x) = x\sqrt{|x^2-1|}$,

v) $p(x) = \operatorname{arctg}(\log(|x^2 - 1|))$, *vi)* $w(x) = 2\operatorname{arctg} x + \arcsin\Big(\dfrac{2x}{1+x^2}\Big)$.

Problem 2.3.5 Draw the graph of the function

$$f(x) = \frac{e^{1/x}}{1+x}, \qquad x \neq 0; \qquad f(0) = 0.$$

and study in a reasoned way how many solution does the equation $f(x) = x^3$ have on $\mathbb{R}$.

Problem 2.3.6 Given the function $f(x) = \dfrac{1+x}{3+x^2}$, represent the graph of the functions

$$i) \quad g(x) = \sup_{y>x} f(y), \qquad ii) \quad h(x) = \inf_{y>x} f(y).$$

Problem 2.3.7

i) Calculate the image of the function $f(x) = 1 + (\operatorname{arctg} x)^2$.

ii) Calculate the values of $A \in \mathbb{R}$ such that the function

$$g(x) = \frac{1}{A + \log f(x)},$$

is continuous on all $\mathbb{R}$.

iii) Find the supremum and infimum of g if $A = 1$.

iv) Sketch the graph of g in this last case.

Problem 2.3.8 We consider the function $f(x) = \log(1 + x^2)$.

i) Calculate the tangent lines at its inflection points, and sketch the graph of f and those lines.

ii) Prove that the function $g(x) = \max\{ f(x), |x| + \alpha \}$ verifies the hypothesis of the Mean Value Theorem on any interval $[a, b] \subset I\!R$ if and only if $\alpha = \log 2 - 1$.

iii) For the previous value of α, obtain the point or points whose existence is guaranteed by the aforementioned theorem applied to function g on the interval $[-1, 2]$.

2.4 Taylor polynomial

Problem 2.4.1 Write the Taylor polynomial of order 5 around the origin for the following functions:

$$i)\quad e^x \sin x, \qquad ii)\quad e^{-x^2} \cos 2x, \qquad iii)\quad \sin x \cos 2x,$$

$$iv)\quad e^x \log(1 - x), \qquad v)\quad (\sin x)^2, \qquad vi)\quad \frac{1}{1 - x^3}.$$

Problem 2.4.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x - 4$.

Problem 2.4.3 Write the Taylor polynomial of order n around the given point, for the following functions:

i) $f(x) = 1/x$ at $a = -1$;

ii) $f(x) = xe^{-2x}$ at $a = 0$;

iii) $f(x) = (1 + e^x)^2$ at $a = 0$.

Problem 2.4.4 Prove the formulas

$$i)\quad \sin x = o(x^\alpha), \quad \forall\, \alpha < 1, \qquad \text{when } x \to 0;$$

$$ii)\quad \log(1 + x^2) = o(x), \qquad \text{when } x \to 0;$$

$$iii)\quad \log x = o(x), \qquad \text{when } x \to \infty;$$

$$iv)\quad \operatorname{tg} x - \sin x = o(x^2), \qquad \text{when } x \to 0.$$

Problem 2.4.5 Prove the formulas

$$i)\quad \sqrt{1 + x} = 1 + \frac{x}{2} + o(x), \qquad \text{when } x \to 0;$$

$$ii)\quad \sin(o(x)) = o(x), \qquad \text{when } x \to 0;$$

$$iii)\quad \sin(f(x) + o(x)) = \sin(f(x)) + o(x), \qquad \text{when } x \to 0.$$

Hint: ii) means that if g is such that $\lim_{x \to 0} \frac{g(x)}{x} = 0$, then $\lim_{x \to 0} \frac{\sin(g(x))}{x} = 0$.

Problem 2.4.6 Find the following limits using Taylor's Theorem:

i) $\lim_{x\to 0} \dfrac{e^x - \sin x - 1}{x^2}$,

ii) $\lim_{x\to 0} \dfrac{\sin x - x + x^3/6}{x^5}$,

iii) $\lim_{x\to 0} \dfrac{\cos x - \sqrt{1-x}}{\sin x}$,

iv) $\lim_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3}$,

v) $\lim_{x\to 0} \dfrac{x - \sin x}{x(1-\cos 3x)}$,

vi) $\lim_{x\to 0} \dfrac{\cos x + e^x - x - 2}{x^3}$,

vii) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$,

viii) $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{x} - \cot x\right)$,

ix) $\lim_{x\to\infty} x^{3/2}(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x})$,

x) $\lim_{x\to\infty} \left[x - x^2 \log(1+1/x)\right]$.

Problem 2.4.7 Calculate the Taylor polynomial of order 4 at the origin for the function $f(x) = 1 + x^3 \sin x$ and decide whether f has at that point a maximum, a minimum or an inflection point.

Problem 2.4.8

i) Calculate approximately the value of $\dfrac{1}{\sqrt{1.1}}$ using a Taylor polynomial of degree 3. How much is the error?

ii) Approximate $\sqrt[3]{28}$ using a Taylor polynomial of degree 2. Evaluate the error.

Problem 2.4.9

i) Approximate the function $f(x) = \cos x + e^x$ through a third order polynomial around the origin.

ii) Estimate the error when the previous approximation is used at $x \in [-1/4, 1/4]$.

Problem 2.4.10 How many terms should be taken in Taylor expansion around the origin for the function $f(x) = e^x$ in order to obtain a polynomial which approximates it on $[-1, 1]$ with three exact significant figures?