UNIVERSIDAD CARLOS III DE MADRID

Escuela Politécnica Superior

Departamento de Matemáticas

PROBLEMS, CALCULUS I, 1st COURSE

3. SEQUENCES AND SERIES

BACHELOR IN:
Audiovisual System Engineering
Communication System Engineering
Telematics Engineering

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3 Sequences and series

3.1 Real numbers sequences

Problem 3.1.1

i) Let $\{x_n\}$ be a convergent sequence and $\{y_n\}$ a divergent one; What can we say about the product sequence $\{x_n y_n\}$, sum sequence $\{x_n + y_n\}$, and quotient sequence $\{y_n/x_n\}$ (if $x_n \neq 0$ for all $n \in I\!N$)?

ii) Prove that if $\{x_n\}$ is convergent, then the sequence $\{|x_n|\}$ is also convergent. Is the reciprocal true?

iii) What can we say about a sequence of integer numbers that is convergent?

iv) Show that every convergent sequence is bounded.

Problem 3.1.2 Given the following sequences in a recurrent way, write down the general term and compute the limit.

$$i)\ a_0 = 0, \quad a_{n+1} = \frac{a_n + 1}{2}; \qquad ii)\ b_0 = 1, \quad b_{n+1} = \sqrt{2b_n}\,.$$

Problem 3.1.3 Compute the following limits:

$$i)\ \lim_{n\to\infty} \sqrt[n]{a}, \quad (a > 0), \qquad ii)\ \lim_{n\to\infty} n^{-3/n},$$

$$iii)\ \lim_{n\to\infty} \sqrt[n]{a^n + b^n}, \quad (a, b > 0), \qquad iv)\ \lim_{n\to\infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n, \quad (a, b > 0),$$

$$v)\ \lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right), \qquad vi)\ \lim_{n\to\infty} (\sqrt[4]{n^2+1} - \sqrt{n+1}),$$

$$vii)\ \lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}, \qquad viii)\ \lim_{n\to\infty} \left(\frac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}.$$

Problem 3.1.4 Calculate the following limits:

$$i)\ \lim_{n\to\infty} \frac{n}{\pi} \sin n\pi, \qquad ii)\ \lim_{n\to\infty} \frac{n(\mathrm{e}^{1/n} - \mathrm{e}^{\sin 1/n})}{1 - n \sin 1/n},$$

$$iii)\ \lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\log n}, \qquad iv)\ \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}},$$

$$v)\ \lim_{n\to\infty} \frac{2^n}{n!}, \qquad vi)\ \lim_{n\to\infty} \frac{n^2}{2^n},$$

$$vii)\ \lim_{n\to\infty} \frac{n^{n-1}}{(n-1)^n}, \qquad viii)\ \lim_{n\to\infty} \frac{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \cdots + n\sqrt[n]{n}}{n^2}.$$

Problem 3.1.5 Compute the following limits:

$$i)\ \lim_{n\to\infty} \left(\cos\frac{b}{n} + a \sin\frac{b}{n}\right)^n; \qquad ii)\ \lim_{n\to\infty} \sqrt[u_n]{\frac{a - bu_n}{a + u_n}}, \quad \text{if } \lim_{n\to\infty} u_n = 0,\ a > 0.$$

Problem 3.1.6 Compute the following limits:

$$i)\ \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\sin\frac{\pi}{k}}{\log n},\qquad ii)\ \lim_{n\to\infty}\prod_{k=1}^{n}(2k-1)^{1/n^2},\qquad iii)\ \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^2}\sin\frac{1}{k}.$$

Problem 3.1.7 If $\lim_{n\to\infty} a_n = \ell$, find

$$\lim_{n\to\infty}\frac{a_1+\frac{a_2}{2}+\cdots+\frac{a_n}{n}}{\log(n+1)}.$$

Problem 3.1.8 Let $\{a_n\}$ be a sequence of positive terms verifying $\lim_{n\to\infty}(a_n-n)=L$.

i) Show that $\lim_{n\to\infty}\frac{a_n}{n}=1$.

ii) Prove that $\lim_{n\to\infty} n\log(a_n/n)=L$.

Problem 3.1.9 Let $\{a_n\}$ be a sequence of positive numbers verifying $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\ell$. Compute, by means of Stolz's test, the limit

$$\lim_{n\to\infty}\sqrt[n^2]{\frac{a_n^n}{a_1\cdot a_2\cdots a_n}}.$$

Problem 3.1.10 Prove the formulas, for $n\to\infty$,

$$i)\ \sin(\pi n+o(1))=o(1),\qquad ii)\ \sin(\pi\sqrt{n^2+1})=(-1)^n\sin\Big(\frac{\pi}{2n}\Big)+o\Big(\frac{1}{n}\Big).$$

Hint: use problem 2.4.5.

Problem 3.1.11 Prove that the following sequences are monotonic, analyze if they are bounded, and compute the limits if they exist.

$$\begin{array}{llll}
i) & \sqrt{2},\sqrt{2\sqrt{2}},\sqrt{2\sqrt{2\sqrt{2}}},\dots & ii) & \sqrt{2},\sqrt{2+\sqrt{2}},\sqrt{2+\sqrt{2+\sqrt{2}}},\dots\\
iii) & u_{n+1}=3+\dfrac{u_n}{2},\quad u_0=0. & iv) & u_{n+1}=3+2u_n,\quad u_0=0.\\
v) & u_{n+1}=\dfrac{u_n^3+6}{7}, & & a)\ u_0=1/2,\quad b)\ u_0=3/2,\quad c)\ u_0=3.
\end{array}$$

Problem 3.1.12 Consider the sequence defined as $a_{n+1}=\sqrt{1+3a_n}-1,\ a_0=1/2$.

i) Prove that it converges and compute its limit.

ii) Compute $\lim_{n\to\infty}\frac{a_{n+1}-1}{a_n-1}$.

Problem 3.1.13 Let the sequence be defined as $b_{n+1}=1-\frac{b_n}{2}$, with $b_0=0$.

i) Show that is an alternating sequence, that is, $\operatorname{sign}(b_{n+1}-b_n)=-\operatorname{sign}(b_n-b_{n-1})$.

ii) Compute the limit ℓ if it exists.

iii) Show that $|b_{n+1} - \ell| = \frac{1}{2}|b_n - \ell|$.

iv) Prove that $\lim_{n\to\infty} b_n = \ell$.

Hint: *iii*) $|b_n - \ell| = (\frac{1}{2})^n\, |b_0 - \ell|$.

Problem 3.1.14 Consider the sequence defined by $c_{n+1} = f(c_n)$, where $f(x) = \dfrac{1}{1+x}$, $c_0 = 0$. Prove that converges by means of the following steps:

i) Compute the limit ℓ if it exists.

ii) Prove that if $x \in [1/2, 1]$, then $f(x) \in [1/2, 1]$.

iii) Prove that $c_n \in [1/2, 1]$ for all $n \geq 1$.

iv) Show that $|f'(x)| \leq k < 1$ for every $x \in [1/2, 1]$. This implies that $|c_{n+1} - \ell| \leq k^n |c_1 - \ell| \to 0$.

Problem 3.1.15

i) Use the technique of the previous problem for the sequence

$$d_0 = \frac{1}{2}, \quad d_{n+1} = 2 + \frac{4}{d_n},\ n \geq 0,$$

on the interval $[3, 10/3]$.

ii) Compute $\lim_{n\to\infty} \dfrac{d_{n+1} - \ell}{d_n - \ell}$.

Problem 3.1.16 Consider the sequence of real numbers defined recurrently

$$x_1 = 1, \qquad x_n = \frac{x_{n-1}(1 + x_{n-1})}{1 + 2x_{n-1}}\,.$$

Prove that converges and compute its limit.

Problem 3.1.17 Describe the behaviour of the recurrent sequences of the previous problems, sketching in two cartesian axes each pair of consecutive terms (*spider's web diagram*).

3.2 Series of real numbers

Problem 3.2.1 Analyze the convergence of the following series of positive terms:

i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)^n$, *ii)* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}$, *iii)* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n^4+1}}$,

iv) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$, *v)* $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2+n}$, *vi)* $\displaystyle\sum_{n=1}^{\infty}\sin(\frac{1}{n^2})$,

vii) $\displaystyle\sum_{n=1}^{\infty}\arcsin(\frac{1}{\sqrt{n}})$, *viii)* $\displaystyle\sum_{n=1}^{\infty}\frac{3n-1}{(\sqrt{2})^n}$, *ix)* $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}$,

x) $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{n}-1)^n$, *xi)* $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}3^{-n}$, *xii)* $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}\mathrm{e}^{-n}$,

xiii) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$, *xiv)* $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{(\log n)^n}$, *xv)* $\displaystyle\sum_{n=2}^{\infty}[\sqrt{n^2+1}-n]$,

xvi) $\displaystyle\sum_{n=2}^{\infty}\log(\frac{n+1}{n})$, *xvii)* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\log n}}$, *xviii)* $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}$.

Hints: (in general it can be applied more than one test); *i*), *viii*), *x*), *xi*), *xiii*), *xiv*), root's test; *ix*), quotient's test; *ii*), *iii*), *iv*), *v*), *vi*), *vii*), *xv*), *xvi*), *xvii*), *xviii*), comparison; *xii*), compute the limit (see 2.1.21 *ii*)).

Problem 3.2.2 Prove that the series

$$\sum_{n=1}^{\infty}\left(\frac{a}{2n-1}-\frac{b}{2n+1}\right)$$

is convergent if and only if $a=b$.

Problem 3.2.3

i) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty}n(1+a)^n\mathrm{e}^{-an}$, for different values of $a>-1$.

ii) Do the same for the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{a^n n!}$, for different values of $a>0$.

iii) Do the same for the series $\displaystyle\sum_{n=1}^{\infty}\frac{n!\mathrm{e}^n}{n^{n+a}}$, for different values of $a\in I\!R$.

Hints: *ii*) and *iii*) use Stirling's formula.

Problem 3.2.4 Analyze the absolute and conditional convergence of the following alternating series:

$$i)\ \sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}, \qquad ii)\ \sum_{n=2}^{\infty}\sin(\pi n+1/n),$$

$$iii)\ \sum_{n=1}^{\infty}(-1)^n(\operatorname{arctg}1/n)^2, \qquad iv)\ \sum_{n=1}^{\infty}(-1)^n(\operatorname{arctg}n)^2,$$

$$v)\ \sum_{n=1}^{\infty}(-1)^n[\sqrt{n^2-1}-n], \qquad vi)\ \sum_{n=1}^{\infty}(-1)^n\log(\frac{n}{n+1}),$$

$$vii)\ \sum_{n=1}^{\infty}(-1)^n(1-\cos(1/n)), \qquad viii)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{\log(\mathrm{e}^n+\mathrm{e}^{-n})}.$$

Problem 3.2.5 Use the Taylor expansion of the function $\operatorname{arctg}x$ to study the convergence of the series

$$\sum_{n=1}^{\infty}\Big(\operatorname{arctg}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n}}\Big).$$

Problem 3.2.6 Compute how many terms are necessary to approximate the following sums with an error less than 10^{-3}:

$$i)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}, \qquad ii)\ \sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Problem 3.2.7 Compute the sum of the following series:

$$i)\ \sum_{n=0}^{\infty}\frac{3^{n+1}-2^{n-3}}{4^n}, \qquad ii)\ \sum_{n=1}^{\infty}\frac{n}{2^n}, \qquad iii)\ \sum_{n=0}^{\infty}\frac{4n+1}{3^n},$$

$$iv)\ \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}, \qquad v)\ \sum_{n=1}^{\infty}\log\left[\frac{n(n+2)}{(n+1)^2}\right].$$

Problem 3.2.8 Compute the sum of the following series:

$$i)\ \sum_{n=0}^{\infty}a^{[n/2]}b^{[(n+1)/2]},\quad (|ab|<1), \qquad ii)\ \sum_{n=1}^{\infty}\frac{1}{2^n}\cos\frac{2n\pi}{3}.$$

Hint: decompose the sums in two or three summands respectively.

Problem 3.2.9

i) Prove that the series $\sum_{n=0}^{\infty}b_n10^{-n}$, where $b_n\in\{0,1,\dots,9\}$ for $n\geq 1$ and $b_0\in\mathbb{Z}$, converges. What represents this series and why is it important?

ii) Compute the previous sum in the cases:

$$a)\ b_n=9,\quad n\geq 0; \qquad b)\ b_n=\begin{cases}1 & n=2k\\ 2 & n=2k+1\end{cases},\quad k\geq 0.$$

Problem 3.2.10 Let $\{a_n\}$ be a sequence of positive terms verifying $\lim_{n\to\infty}(a_n - n) = L > 0$.

i) Prove that the series $\sum_{n=1}^{\infty} n^{\alpha}\log(a_n/n)$ converges if and only if $\alpha < 0$.

ii) Compute the limits

$$P = \lim_{n\to\infty}\Big[\prod_{i=1}^{n}(\frac{a_i}{i})\Big]^{1/n}, \qquad Q = \lim_{n\to\infty}\Big[\prod_{i=1}^{n}(\frac{a_i}{i})^i\Big]^{1/n}.$$

Hint: use problem 3.1.8.

Problem 3.2.11

i) Prove that the equation $\operatorname{tg} x = x$ has an unique solution λ_n on each interval $((2n-1)\pi/2, (2n+1)\pi/2)$, $n = 1, 2, 3, \cdots$

ii) Prove that the series $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}$ is convergent.

Problem 3.2.12 Study the convergence of the series

$$a) \quad \sum_{n=1}^{\infty}\sin(\pi n(1+\frac{1}{2n^2})), \qquad b) \quad \sum_{n=1}^{\infty}\sin^2(\pi\sqrt{n^2+1}).$$

Hint: use problem 3.1.10.

Problem 3.2.13 Let the sequence defined as $x_{n+1} = \sqrt{1+2x_n} - 1$, $x_0 = 1$.

i) Prove that is convergent and compute its limit.

ii) Compute the limits

$$a) \quad \lim_{n\to\infty}\frac{x_{n+1}}{x_n}, \qquad b) \quad \lim_{n\to\infty} n x_n.$$

iii) Study the convergence of the series

$$a) \quad \sum_{n=0}^{\infty} x_n, \qquad b) \quad \sum_{n=0}^{\infty} x_n^2.$$

Hint: *ii.b*) apply Stolz conveniently; *iii*) apply part *ii*).

3.3 Taylor series

Problem 3.3.1 Find the interval of convergence of the following power series:

$$i) \quad \sum_{n=1}^{\infty}\frac{x^n}{2^n n^2}, \qquad ii) \quad \sum_{n=1}^{\infty}\frac{n!x^n}{n^n}, \qquad iii) \quad \sum_{n=1}^{\infty}\frac{x^n}{n10^{n-1}},$$

$$iv) \quad \sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}, \qquad v) \quad \sum_{n=1}^{\infty}(3-2x)^n, \qquad vi) \quad \sum_{n=1}^{\infty}\frac{(x-2)^n}{\sqrt{2n}}.$$

Problem 3.3.2 Expand in power series the function $f_k(x) = \dfrac{1}{(1-x)^k}$, for $k = 1, 2, 3$.

Problem 3.3.3 Compute the radius of convergence and the sum of the following power series:

$$i) \quad \sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad ii) \quad \sum_{n=0}^{\infty} (n+1)2^{-n}x^n.$$

Problem 3.3.4 Expand in power series, showing the domain where the series is valid, of the functions:

$$i) \quad f(x) = \sin^2 x, \qquad ii) \quad f(x) = \frac{x}{a+bx}, \quad \text{with } a, b > 0,$$

$$iii) \quad f(x) = \log\sqrt{\frac{1+x}{1-x}}, \qquad iv) \quad f(x) = \frac{1}{2-x^2}, \qquad v) \; f(x) = \mathrm{e}^{x^2}.$$

Problem 3.3.5 Compute the sum of the following series

$$i) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!}, \qquad ii) \quad \sum_{n=1}^{\infty} \frac{n}{2^n},$$

$$iii) \quad \sum_{n=1}^{\infty} \frac{1}{n2^n}, \qquad iv) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}.$$

Problem 3.3.6 Let C_0 be a circle of radius r.

i) Obtain a rectangle Q_0, inscribed in C_0, of maximum area.

ii) Let now C_1 be the maximum interior circle to that rectangle, concentric with C_0, and inscribe a rectangle Q_1 of maximum area in C_1. Compute the sum of the areas of the sequence $\{C_n\}_{n=0}^{\infty}$ of circles obtained by iterating the process.

Problem 3.3.7 Given the function $f(x) = \sum_{n=1}^{\infty} \frac{n^x}{n!}$, compute the values of $f(0)$, $f(1)$ and $f(2)$.

Problem 3.3.8 Find a function $f(x)$, with power series expansion, verifying

$$f'(x) = f(x) + x, \qquad f(0) = 2.$$