UNIVERSIDAD CARLOS III DE MADRID

Escuela Politécnica Superior

Departamento de Matemáticas

PROBLEMS, CALCULUS I, 1st COURSE

4. INTEGRATION IN ONE VARIABLE

BACHELOR IN:
Audiovisual System Engineering
Communication System Engineering
Telematics Engineering

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4 Integration in one variable

4.1 Antiderivatives

Problem 4.1.1 Find the following antiderivatives:

1. $\int x \operatorname{tg}^2(2x)\,dx$,
2. $\int \operatorname{tg}^3 x \sec^4 x\,dx$,
3. $\int \frac{\sqrt{x}+1}{x+3}\,dx$,
4. $\int \frac{(x+3)^3}{\sqrt{1-(x+1)^2}}\,dx$,
5. $\int \frac{x^2}{(x-1)^3}\,dx$,
6. $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$,
7. $\int \frac{\sin^2 x \cos^5 x}{\operatorname{tg}^3 x}\,dx$,
8. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$,
9. $\int \mathrm{e}^x \sin \pi x\,dx$,
10. $\int \frac{dx}{\cos^4 x}$,
11. $\int \sin^2 x\,dx$,
12. $\int \sin^4 x\,dx$,
13. $\int \cos^2 x\,dx$,
14. $\int \cos^6 x\,dx$,
15. $\int \sin^2 x \cos^2 x\,dx$,
16. $\int \frac{dx}{3+\sqrt{2x+5}}$,
17. $\int \sqrt{\frac{x-1}{x+1}}\,dx$,
18. $\int \operatorname{arctg} \sqrt[3]{x}\,dx$,
19. $\int \sqrt{\sqrt{x}+1}\,dx$,
20. $\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$,
21. $\int \sqrt{2+\mathrm{e}^x}\,dx$,
22. $\int \mathrm{e}^{\sin x} \cos^3 x\,dx$,
23. $\int \sin^5 x\,dx$,
24. $\int \cos^3 x \sin^2 x\,dx$,
25. $\int \operatorname{tg}^2 x\,dx$,
26. $\int \operatorname{tg}^3 x\,dx$,
27. $\int x^3 \sqrt{1-x^2}\,dx$,
28. $\int \frac{\sin x + 3\cos x}{\sin x \cos x + 2\sin x}\,dx$,
29. $\int \frac{\sin x + 3\cos x}{\sin x + 2\cos x}\,dx$,
30. $\int \operatorname{tg}^2(3x) \sec^3(3x)\,dx$,
31. $\int \frac{4x^4 - x^3 - 46x^2 - 20x + 153}{x^3 - 2x^2 - 9x + 18}\,dx$,
32. $\int \cos(\log x)\,dx$,
33. $\int \frac{\mathrm{e}^{4x}}{\mathrm{e}^{2x}+\mathrm{e}^x+2}\,dx$,
34. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$,
35. $\int \frac{x^2}{(x^2+1)^{5/3}}\,dx$,
36. $\int \frac{2}{x^2-2x+2}\,dx$,
37. $\int \frac{dx}{\cos^2 x}$,
38. $\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$,
39. $\int \frac{x}{(x^2+1)^{5/2}}\,dx$,
40. $\int x^2(1-x^2)^{-3/2}\,dx$,
41. $\int \sqrt{\mathrm{e}^x - 1}\,dx$,
42. $\int \frac{2x^2+3}{x^2(x-1)}\,dx$,

43. $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$,
44. $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$,
45. $\displaystyle\int x^2\sqrt{x-1}\,dx$,
46. $\displaystyle\int \sec^6 x\,dx$,
47. $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$,
48. $\displaystyle\int \frac{dx}{e^x-4e^{-x}}$,
49. $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$,
50. $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$,
51. $\displaystyle\int e^x\cos 2x\,dx$,
52. $\displaystyle\int x^2\log x\,dx$,
53. $\displaystyle\int \sin^3 x\cos^2 x\,dx$,
54. $\displaystyle\int \cos^4 x\,dx$,
55. $\displaystyle\int \mathrm{tg}^4 x\,dx$,
56. $\displaystyle\int \sec^3 x\,dx$,
57. $\displaystyle\int \frac{dx}{1-\sin x}$,
58. $\displaystyle\int \sin(\log x)dx$,
59. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$,
60. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$,
61. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$,
62. $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$,
63. $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$,
64. $\displaystyle\int \frac{dx}{\sqrt[3]{(1-2x)^2}-\sqrt{1-2x}}$,
65. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$,
66. $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$,
67. $\displaystyle\int x^m\log x\,dx$,
68. $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$,
69. $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$,
70. $\displaystyle\int \cos^2(\log x)\,dx$,
71. $\displaystyle\int (\log x)^3\,dx$,
72. $\displaystyle\int x(\log x)^2\,dx$.

Hint: IBP means integration by parts and CV change of variables.

1. IBP $dv=\mathrm{tg}^2(2x)dx$.
2. CV $t=\mathrm{tg}\ x$.
3. CV $t=\sqrt{x}$.
4. CV $t=\sqrt{1-(x+1)^2}$.
5. Partial fraction decomposition.
6. CV $x=\sec t$.
7. CV $t=\cos x$.
8. The derivative of the denominator almost appears in the numerator.
9. IBP twice using $dv=e^x dx$.
10. CV $t=\mathrm{tg}\, x$.
11, 12, 13, 14 and 15. Use the double angle formulas.
16. CV $t=\sqrt{2x+5}$.
17. CV $t=\sqrt{(x-1)/(x+1)}$.
18. CV $x=t^3$, after do IBP with $dv=t^2dt$.
19. CV $t=\sqrt{\sqrt{x}+1}$.
20. CV $t=\sqrt{x+2}$.
21. CV $t=\sqrt{e^x+2}$.
22. CV $t=\sin x$, after do IBP twice with $dv=e^t dt$.
23. CV $t=\cos x$.
24. CV $t=\sin x$.
25. $\mathrm{tg}^2 x=\sec^2 x-1$.
26. CV $t=\mathrm{tg}\, x$.
27. CV $t=\sqrt{1-x^2}$.
28 and 29. CV $t=\mathrm{tg}(x/2)$.
30. CV $t=\sin(3x)$.
31. Partial fraction decomposition.
32. IBP twice using $dv=dx$.
33. CV $t=e^x$.
34. CV $t=\sqrt{1+x^{1/3}}$.
35. CV $x=\mathrm{tg}\ t$.
36. Complete the square.

37. It is immediate.
38. CV $x+2=t^3$.
39. CV $t=(x^2+1)^{1/2}$.
40. CV $x=\sin t$.
41. CV $t=\sqrt{\mathrm{e}^x-1}$.
42. Partial fraction decomposition.
43. CV $t=\sqrt{1-\sqrt{x}}$.
44. Multiply and divide by $1-\cos x$.
45. CV $t=\sqrt{x-1}$.
46. CV $t=\operatorname{tg} x$.
47. CV $t=1+x^2$.
48. CV $t=\mathrm{e}^x$.
49. CV $t^2=1+x$.
50. CV $t^3=1-x$.
51. IBP twice using $dv=\mathrm{e}^x dx$.
52. IBP $dv=x^2dx$.
53. CV $t=\cos x$.
54. Use the double angle formulas.
55. CV $t=\operatorname{tg} x$.
56. CV $t=\sin x$.
57. Multiply and divide by $1+\sin x$.
58. CV $t=\log x$.
59. CV $t=\sin x$.
60. CV $t^2=1+x^2$.
61. CV $t^2=\mathrm{e}^{2x}-1$.
62. CV $t=\mathrm{e}^x$.
63. Partial fraction decomposition.
64. CV $t^2=1-2x$.
65. CV $x=3\sin t$.
66. Complete the square.
67. IBP $dv=x^3x$.
68. CV $t=\sin x$.
69. CV $t^2=x^3$.
70. CV $t=\log x$ and use the double angle formulas.
71. IBP $dv=dx$.
72. IBP $dv=xdx$.

Problem 4.1.2 Find a continuous function f such that $f(0)=0$ and

$$f'(x)=\begin{cases} \dfrac{4-x^2}{(4+x^2)^2} & x<0 \\ e^{\sqrt{x}} & x>0. \end{cases}$$

Problem 4.1.3 Compute $\displaystyle\int_a^b x\,dx$ using upper and lower sums associated to regular partitions of the interval $[a,b]$.

Problem 4.1.4

i) Prove that, if g is an odd and integrable function on $[-a,a]$, then $\displaystyle\int_{-a}^a g=0$. Apply the result to compute

$$\int_6^{10} \sin[\sin\{(x-8)^3\}]dx\,.$$

ii) Prove that, if h is an even and integrable function on $[-a,a]$, then $\displaystyle\int_{-a}^a h=2\int_0^a h$.

Problem 4.1.5 Prove and interpret the following identities:

$$i) \qquad \int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx,$$

$$ii) \qquad \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx,$$

$$iii) \qquad \int_{-a}^{a} [f(x) - f(-x)]\,dx = 0,$$

$$iv) \qquad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx,$$

$$v) \qquad \int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^{ab} \frac{dx}{x}.$$

Problem 4.1.6 Let f be a periodic function of period T, integrable on $[0,T]$. Prove that:

i) for all integer n, we have

$$\int_a^b f = \int_{a+nT}^{b+nT} f;$$

ii) for all $a \in [0,T)$, we have

$$\int_a^{a+T} f = \int_0^T f;$$

Problem 4.1.7 Evaluate the following limits associating them to some definite integral:

$$i) \qquad \lim_{n\to\infty} \left[\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right],$$

$$ii) \qquad \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right],$$

$$iii) \qquad \lim_{n\to\infty} \frac{\sqrt[n]{e^2} + \sqrt[n]{e^4} + \cdots + \sqrt[n]{e^{2n}}}{n},$$

$$iv) \qquad \lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right].$$

Problem 4.1.8 Compute the limit

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k}{n}\right)^{1/n}.$$

Problem 4.1.9 Evaluate $F(x) = \int_{-1}^{x} f(t)\,dt$ with $x \in [-1,1]$, for the following functions:

i) $f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x \le 1; \end{cases}$

ii) $f(x) = |x|\, e^{-|x|};$

iii) $f(x) = |x - 1/2|;$

iv) $f(x) = \begin{cases} x^2 & -1 \le x < 0 \\ x^2 - 1 & 0 \le x \le 1; \end{cases}$

v) $f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ x+1 & 0 < x \le 1; \end{cases}$

vi) $f(x) = \begin{cases} x+2 & -2 \le x \le -1 \\ 1 & -1 < x < 1 \\ -x+2 & 1 \le x \le 2; \end{cases}$

vii) $f(x) = \max\{\sin(\pi x/2), \cos(\pi x/2)\}.$

Problem 4.1.10 Compute the following definite integrals, changing the limits of integration when making a change of variables:

$$i)\quad \int_0^{\log 2} \sqrt{\mathrm{e}^x - 1}\,dx, \qquad ii)\quad \int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx.$$

4.2 The Fundamental Theorem of Calculus

Problem 4.2.1 Let $F(x) = \int_a^x f(t)\,dt$ with f integrable.

i) Prove that if $|f| \le M$ then $|F(x) - F(y)| \le M|x-y|$, implying the continuity of F.

ii) Is F differentiable necessarily? Under what conditions can we say that is differentiable?

Problem 4.2.2 Differentiate the following functions:

i) $F(x) = \int_{x^2}^{x^3} \frac{\mathrm{e}^t}{t}\,dt,$

ii) $F(x) = \int_{-x^3}^{x^3} \frac{dt}{1+\sin^2 t},$

iii) $F(x) = \int_3^{\int_1^x \sin^3 t dt} \frac{dt}{1+\sin^6 t + t^2},$

iv) $F(x) = \int_2^{\mathrm{e}^{\int_1^{x^2} \mathrm{tg}\sqrt{t}\,dt}} \frac{ds}{\log s}$

$$v)\quad F(x) = \int_0^x x^2 f(t)dt, \quad \text{where } f \text{ is continuous on } \mathbb{R},$$

$$vi)\quad F(x) = \sin\left(\int_0^x \sin\left(\int_0^y \sin^3 t dt\right) dy\right).$$

Problem 4.2.3 Compute the maximum and the minimum on $[1,\infty)$ of the function:

$$f(x) = \int_0^{x-1} \left(\mathrm{e}^{-t^2} - \mathrm{e}^{-2t}\right)\,dt.$$

Problem 4.2.4

i) Show that the equation

$$\int_0^x e^{t^2}\,dt = 1$$

has a unique solution on $\mathbb{R}$ and that such solution lies on the interval $(0,1)$.

ii) Find and classify the local maxima and minima on $(0,\infty)$ of the function

$$G(x) = \int_0^{x^2} \sin t\, e^{\sin t}\,dt\,.$$

Problem 4.2.5 Find the tangent line to the graph of $y = \displaystyle\int_{x^2}^{\sqrt{\pi/2}} \operatorname{tg}(t^2)\,dt$ at $x = \sqrt[4]{\pi/4}$.

Problem 4.2.6 Calculate the following limits:

$$i)\quad \lim_{x\to 0} \frac{\displaystyle\int_0^x e^{t^2}\,dt - x}{x^3}, \qquad ii)\quad \lim_{x\to 0} \frac{\cos x \displaystyle\int_0^x \sin t^3\,dt}{x^4}.$$

Problem 4.2.7 Compute one-sided limits at the origin of the function

$$f(x) = \frac{x - \sin x + \displaystyle\int_0^{x^2} \operatorname{tg}(\sqrt{t})\,dt}{2x^3}\,.$$

Problem 4.2.8 Consider the function $f(x) = \displaystyle\int_0^{x^2} \frac{\sin t}{t}\,dt$.

i) Using the Taylor series of the sine function, find the Taylor series of f around the origin.

ii) Compute $\displaystyle\lim_{x\to 0} \frac{f(x)}{1-\cos x}$.

iii) Analyze the convergence of the series $\displaystyle\sum_{n=1}^{\infty} f(1/n)$.

Problem 4.2.9 If the integral $\displaystyle\int_{-1/x}^{x} \frac{dt}{a^2+t^2}$ does not depend on x, without computing the integral, find a.

Problem 4.2.10 Consider the functions $f(x) = e^{x^2} - x^2 - 1$, $g(x) = 3 + \displaystyle\int_0^x f(t)\,dt$.

i) Write down the Taylor polynomial of g around the origin.

ii) Determine if g has a maximum, a minimum or an inflection point around the origin.

Problem 4.2.11 Let g be a derivable function verifying the equation

$$t = \int_0^{(g(t))^2} \frac{\sin x}{x}\,dx.$$

i) Write down $g'(t)$ in terms of $g(t)$.

ii) Compute $(g^{-1})'(x)$.

Problem 4.2.12 The equation

$$\int_0^{g(x)} (\mathrm{e}^{t^2} + \mathrm{e}^{-t^2})\, dt - x^3 - 3 \operatorname{arctg} x = 0,$$

defines a differentiable and one to one function g differentiable on $I\!R$. Compute:

i) $g(0)$, $g'(0)$ and $(g^{-1})'(0)$;

ii) $\lim_{x\to 0} \dfrac{g^{-1}(x)}{g(x)}$.

Problem 4.2.13 Find the explicit formula of a continuous function, $f : I\!R \to I\!R$, verifying

$$\int_0^x f(t)\, dt = \int_x^1 t^2 f(t)\, dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C.$$

Next, find the value of C.

4.3 Applications

Problem 4.3.1 Find the area enclosed by the following curves:

i) $y = x^2,\ y = (x-2)^2,\ y = (2-x)/6$;

ii) $x^2 + y^2 = 1,\ x^2 + y^2 = 2x$;

iii) $y = \dfrac{1-x}{1+x},\ y = \dfrac{2-x}{1+x},\ y = 0,\ y = 1$;

iv) loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

Problem 4.3.2 Find the area bounded by the graph of $f(x) = \dfrac{x(x^2-1)}{(x^2+1)^{3/2}}$ and the horizontal axis.

Problem 4.3.3 Find the area enclosed by the following curves given in parametric and polar coordinates:

i) loop: $x = t^2 + 1,\ y = t(t^2 - 4), \quad -2 \le t \le 2$;

ii) cycloid: $x = a(t - \sin t),\ y = a(1 - \cos t), \quad 0 \le t \le 2\pi$, and axis X;

iii) spiral of Archimedes: $r = a\theta, \quad 0 \le \theta \le 2\pi$ and the line segment $\{0 \le x \le 2\pi a,\ y = 0\}$;

iv) one leaf of the three-leaved rose: $r = a\cos 3\theta, \quad -\pi/6 \le \theta \le \pi/6$;

v) half of the lemniscate: $r = a\sqrt{\cos 2\theta}, \quad -\pi/4 \le \theta \le \pi/4$.

Problem 4.3.4

i) Find the area between the graph of the function $f(x) = \dfrac{x^2-4}{x^2+4}$ and its asymptote.

ii) Find the area enclosed by the graph of the function $f(x) = \dfrac{1}{1+\mathrm{e}^{-x}}$, its asymptote at $x \to +\infty$ and the vertical axis.

iii) Compute the area enclosed by the graph of the function $f(x) = \dfrac{1-x}{(x+1)^2\sqrt{x}}$ and its asymptotes.

iv) Find the area enclosed by the graphs of the functions $f_1(x) = \dfrac{x-4}{(x+4)\sqrt{x}}$ and $f_2(x) = \dfrac{1}{\sqrt{x}}$ for $x \geq 4$.

Problem 4.3.5 Let A be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Compute the area of A and the revolution volume obtained by rotating A about the horizontal axis.

Problem 4.3.6 Evaluate the volumes formed by revolving the following regions about the X axis:

i) $0 \leq y \leq 1 + \sin x, \quad 0 \leq x \leq 2\pi;$

ii) $x^2 + (y-2a)^2 \leq a^2$ (the graph is a torus);

iii) $R^2 \leq x^2 + y^2 \leq 4R^2$ (an spherical ring);

iv) the surface bounded by the curves $y = \sin x$ and $y = x$, with $x \in [0, \pi]$;

v) $x = t - \sin t, \ 0 \leq y \leq 1 - \cos t, \quad 0 \leq t \leq 2\pi.$

Problem 4.3.7

i) Compute the volumes formed by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1$ about the X and Y axes.

ii) Compute the volume of the solid with base the previous ellipse and whose perpendicular sections to the OX axis are isosceles triangles of height 2.

Problem 4.3.8

i) Compute the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leq 1.$

ii) Compute the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1.$

iii) Show that the previous problem (part *i*)) is a particular instance of this one.

Hint: observe that when cutting the ellipsoid in parallel sections to the coordinate planes, we obtain ellipses.

Problem 4.3.9 Find the length of the following graphs:

i) catenary: $y = e^{x/2} + e^{-x/2}, \quad 0 \leq x \leq 2;$

ii) cycloid: $x(t) = a(t - \sin t), \ y(t) = a(1 - \cos t), \quad 0 \leq t \leq 2\pi;$

iii) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4;$

iv) tractrix: $y = a \log\Big(\dfrac{a + \sqrt{a^2 - x^2}}{x}\Big) - \sqrt{a^2 - x^2}, \quad a/2 \leq x \leq a;$

v) cardioid: $r = 1 + \cos\theta, \quad 0 \leq \theta \leq 2\pi;$

vi) circular helix: $x(t) = a\cos t, \ y(t) = a\sin t, \ z(t) = bt, \quad 0 \leq t \leq 2\pi.$