# Universidad Carlos III de Madrid

## Escuela Politécnica Superior

Departamento de Matemáticas
First Course. Bachelor's Degree in
Telematics, Communication System and Audiovisual System Engineering

GRADE

**Calculus I.** Second Test, November 25, 2008

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Surname.............................................................. Name...............................
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**Time length: 80 min.**

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**1. (Class problem)** Approximate $e$ using a Taylor Polynomial, with error less than 0.001. How many terms must we use in the polynomial?

[2 p.]

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**2. (Problem 3.1.4)** Compute the limit

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\log n}$$

[2 p.]

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**3. (Problems 3.2.1, 3.2.4)** Analyze the convergence of the following series

1. $\displaystyle\sum_{n=2}^{\infty} \log\left(\frac{n+1}{n}\right)$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\arctan\frac{1}{n}\right)^2$

Hint: $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)}$. [4 p.]

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**4. (Problem 3.3.4)** Compute the Taylor Series, the radius and the interval of convergence of

$$f(x) = \frac{1}{2-x^2}$$

Hint: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n \cdots$ .

[2 p.]

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**ANSWERS:**

**1.** $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + E, \quad E(x) = \dfrac{e^z}{(n+1)!}x^{n+1}, \quad z \in (0, x),$

$E(1) = \dfrac{e^z}{(n+1)!} < \dfrac{3}{(n+1)!} < 0.001 \Rightarrow 3000 < (n+1)! \Rightarrow n = 6$

$e \approx 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} = 2.718.$

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**2.** $$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\log n} \underset{Stolz}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\log\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\log(1 + 1/n)} = 1.$$

$$\lim_{x\to\infty} \frac{\frac{1}{x+1}}{\log(1 + 1/x)} \underset{Taylor}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{1/x} = \lim_{x\to\infty} \frac{x}{x+1} = 1.$$

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**3.**

1. By the limit comparison test: $\lim_{n\to\infty} \dfrac{\log\left(\frac{n+1}{n}\right)}{1/n} = \lim_{n\to\infty} \log(1 + 1/n)^n = \log e = 1 \Rightarrow$ the series diverges.
2. As $(\arctan 1/n)^2 = \left(\dfrac{1}{n} + \circ\left(\dfrac{1}{n^2}\right)\right)^2 = \dfrac{1}{n^2} + \circ\left(\dfrac{1}{n^2}\right)$, by the limit comparison test, the series is absolutely convergent:

   $$\lim_{n\to\infty} \frac{(\arctan n)^2}{1/n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2} + \circ\left(\frac{1}{n^2}\right)}{1/n^2} = 1.$$

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**4.** $f(x) = \dfrac{1}{2 - x^2} = \dfrac{1}{2} \cdot \dfrac{1}{1 - x^2/2} = \dfrac{1}{2}\left(1 + \dfrac{x^2}{2} + \left(\dfrac{x^2}{2}\right)^2 + \left(\dfrac{x^2}{2}\right)^3 + \cdots\right) = \dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2^n},$

hence $a_n = \dfrac{1}{2^{n/2}} \Rightarrow \dfrac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{\dfrac{1}{2^{n/2}}} = 1/\sqrt{2}$ so the radius of convergence is $\rho = \sqrt{2}$.

To study the interval of convergence, we check the end points, as $f(\pm\sqrt{2}) = 1 + 1 + 1 + 1 + \cdots$ is divergent, we conclude that the interval of convergence is $(-\sqrt{2}, \sqrt{2})$.

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