Universidad Carlos III de Madrid
Escuela Politécnica Superior
Departamento de Matemáticas

First Course. Bachelor's Degree in
Telematics Engineering, Communication System Engineering and Audiovisual System Engineering

CALCULUS I   Final Exam, January 22nd 2009.

Time length: 3.30 hours

Problem 1 (3 p.) Consider the function defined as:

$$f(x) = \begin{cases} \ln(x^2) & x \in (-\infty, -1) \cup (1, \infty) \\ -x^2 + 1 & x \in [-1, 1] \end{cases}$$

a) Analyze its continuity and differentiability.

b) Compute its global and local maxima and minima.

c) Sketch its graph.

Problem 2 (1 p.) Let $\{a_n\}$ be a sequence of positive terms verifying $\lim_{n\to\infty} a_n = L$, compute the limit

$$\lim_{n\to\infty} \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}.$$

Problem 3 (2 p.)

a) Study if the following series converges or not

$$\sum_{n=1}^{\infty} \frac{e^{2n}\sqrt{n}}{(n!)^2}.$$

b) Sum the series

$$\sum_{n=1}^{\infty} \left(\frac{\sqrt{n}}{e^n} - \frac{\sqrt{n+1}}{e^{n+1}} \right).$$

Problem 4 (2 p.) Let $f(x)$ be the function defined as

$$f(x) = \int_0^{x^2} \frac{\log(1+t)}{t}\, dt$$

a) Compute the Taylor series of $f(x)$ around $x = 0$ using the Taylor series of $\log(1+t)$.

b) Compute the Taylor series of $f'(x)$ around $x = 0$. Analyze whether $x = 0$ is a local maximum or minimum of $f(x)$.

Problem 5 (2 p.) Find the following antiderivatives

$$\int \frac{dx}{x^4 + 2x^2}, \qquad \int_1^{e^2} \log(\sqrt{x})\, dx.$$

Problem 1

a) Continuity: As $\ln(x^2)$ and $-x^2+1$ are continuous on their domains, we have to check the continuity only on $\{-1,1\}$:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) = 0$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = f(-1) = 0$$

Therefore, $f(x)$ is continuous on $\mathbb{R}$.

Differentiability:

$$f'(x) = \begin{cases} 2/x & x \in (-\infty,-1)\cup(1,\infty) \\ -2x & x \in (-1,1) \end{cases}$$

$$\lim_{x\to 1^+} f'(x) = 2 \neq \lim_{x\to 1^-} f'(x) = -2$$
$$\lim_{x\to -1^+} f'(x) = 2 \neq \lim_{x\to -1^-} f'(x) = -2$$

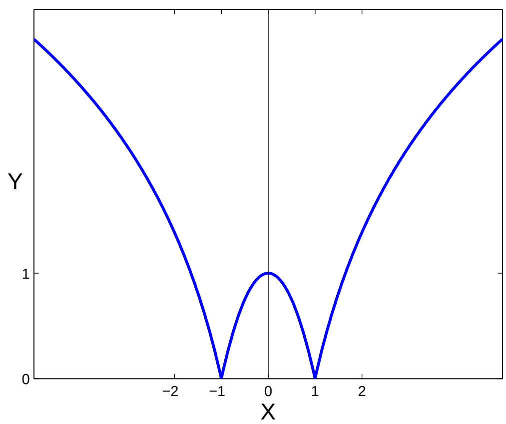
Therefore, $f(x)$ is differentiable on $\mathbb{R}-\{-1,1\}$.

b) The critical points of f are the points in the domain where the function is not differentiable, that is, $\{-1,1\}$, and the points whose derivative vanishes, $f'(x)=0 \Rightarrow x=0$. So we have to study the points $\{-1,0,1\}$. Using the first derivative test, we obtain

x	$f'(x)$	
-1	$- \to +$	local minimum
0	$+ \to -$	local maximum
1	$- \to +$	local minimum

Since $f(-1)=f(1)=0$ and $f(x)\geq 0$, these points are global minima. As $\lim_{x\to\infty} f(x)=\infty$ there is no global maximum.

c) The graph of f is



Problem 2

$$\lim_{n\to\infty}\frac{a_1^2+a_2^2+\cdots+a_n^2}{n} \underset{Stolz}{=} \lim_{n\to\infty}\frac{a_1^2+a_2^2+\cdots+a_n^2-(a_1^2+a_2^2+\cdots+a_{n-1}^2)}{n-(n-1)} = \lim_{n\to\infty} a_n^2 = L^2.$$

Problem 3

a) By the quotient test, it converges:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{e^{2n+2}\sqrt{n+1}}{\big((n+1)!\big)^2}\frac{(n!)^2}{e^{2n}\sqrt{n}} = \lim_{n\to\infty}\frac{e^2}{(n+1)^2}\frac{\sqrt{n+1}}{\sqrt{n}} = 0 < 1.$$

b) It is a telescoping series, therefore,

$$\sum_{n=1}^{\infty}\left(\frac{\sqrt{n}}{e^n}-\frac{\sqrt{n+1}}{e^{n+1}}\right) = \frac{1}{e} - \lim_{n\to\infty}\frac{\sqrt{n}}{e^n} = \frac{1}{e}.$$

Problem 4

a)

$$\begin{aligned} f(x) &= \int_0^{x^2}\frac{\log(1+t)}{t}dt = \int_0^{x^2}\frac{t-t^2/2+\cdots+(-1)^{n+1}t^n/n+\cdots}{t}\,dt = \\ &= \int_0^{x^2}(1-t/2+\cdots+(-1)^{n+1}t^{n-1}/n+\cdots)\,dt = \\ &= \left[t-\frac{t^2}{2\cdot 2}+\frac{t^3}{3\cdot 3}+\cdots+(-1)^{n+1}\frac{t^n}{n^2}+\cdots\right]_0^{x^2} = \\ &= x^2-\frac{x^4}{4}+\frac{x^6}{9}+\cdots+(-1)^{n+1}\frac{x^{2n}}{n^2}+\cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n^2}. \end{aligned}$$

b)

$$f'(x) = 2x - x^3 + \frac{6}{9}x^5 + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}x^{2n-1}.$$

$f(0)$ is a local minimum.

Problem 5 In the first integral we must do partial fraction decomposition:

$$\frac{1}{x^2(x^2+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2} \to \; A=C=0,\; B=1/2,\; D=-1/2 \to$$

$$\int\frac{dx}{x^4+2x^2} = \int\frac{dx}{x^2(x^2+2)} = \frac{1}{2}\int\frac{dx}{x^2} - \frac{1}{2}\int\frac{dx}{x^2+2} = \frac{1}{2x} - \frac{1}{2\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + c.$$

In the second integral we change variables first, $\sqrt{x}=t$, and then we integrate by parts using $u=\log t$ and $dv=2tdt$, hence the integral is

$$\int_1^{e^2}\log\sqrt{x}\,dx = \int_1^e \log t\cdot 2t\,dt = \log t\cdot t^2\Big|_1^e - \int_1^e t\,dt = e^2 - \frac{e^2}{2}+\frac{1}{2} = \frac{e^2}{2}+\frac{1}{2}.$$
