Universidad Carlos III de Madrid
Escuela Politécnica Superior
DEPARTAMENTO DE MATEMÁTICAS

First Course. Bachelor's Degree in
Telematics Engineering, Communication System Engineering and Audiovisual System Engineering

CALCULUS I   Final Exam, June 24th 2009.

Time length: 3.30 hours

Problem 1 (2 p.)

a) Sketch the graph of the function $f(x) = \sqrt{x^2 - 4x + 3}$, analyze the domain, continuity, differentiability, increasing and decreasing intervals, concavity and convexity, extrema and asymptotes.

b) Do the same for the function $g(x) = \sqrt{|x^2 - 4x + 3|}$.

Problem 2 (2 p.) Let the recurrent sequence be defined as

$$b_0 = 10\,, \qquad b_{n+1} = 2 + \frac{b_n}{3}\,,\ n \geq 0.$$

a) Prove that is monotonic and bounded.

b) Compute $\lim_{n\to\infty} b_n$.

Problem 3 (2 p.) Let $g(x)$ be the function defined as $g(x) = \int_0^{x^2} \frac{\cos t - 1}{t}\,dt$.

a) Compute the Taylor series of $g(x)$ around $x = 0$ using the Taylor series of $h(t) = \cos t$.

b) Compute $\lim_{x\to 0} \frac{g(x)}{x^4}$.

Problem 4 (2 p.)

a) Study if the following series converge or not:

$$i)\quad \sum_{n=1}^{\infty} \frac{(n+2)^4}{n!}\,, \qquad ii)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\log n}\,.$$

b) Sum the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}}{2^{2n} n!}$.

Problem 5 (2 p.)

a) Find the following antiderivative $\int \operatorname{arctg} x\,dx$.

b) Compute the volume of revolution obtained by rotating

$$\{0 \leq x \leq 3,\ 0 \leq y \leq \frac{x}{\sqrt{25 - x^2}}\}$$

about the x-axis.

Problem 1

a) The radicand is negative on $(1,3)$, thus the domain is $\mathbb{R}\setminus(1,3)$. As it is the root of a polynomial, it is continuous on its domain and differentiable on $\mathbb{R}\setminus[1,3]$.
Let as see the asymptotes:

$$\lim_{x\to 1^-}\sqrt{x^2-4x+3}=0, \qquad \lim_{x\to 3^+}\sqrt{x^2-4x+3}=0,$$

hence there are no vertical asymptotes.

$$\lim_{x\to\infty}\sqrt{x^2-4x+3}=\infty, \qquad \lim_{x\to-\infty}\sqrt{x^2-4x+3}=\infty,$$

there are no horizontal asymptotes.

$$\lim_{x\to\infty}\frac{\sqrt{x^2-4x+3}}{x}=1, \qquad \lim_{x\to\infty}\left[\sqrt{x^2-4x+3}-x\right]=\lim_{x\to\infty}\frac{-4x+3}{\sqrt{x^2-4x+3}+x}=-2,$$

therefore, $y=x-2$ is an oblique asymptote for $x\to\infty$.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2-4x+3}}{x}=-1, \qquad \lim_{x\to-\infty}\left[\sqrt{x^2-4x+3}+x\right]=\lim_{x\to\infty}\frac{-4x+3}{\sqrt{x^2-4x+3}-x}=2,$$

therefore, $y=-x+2$ is an oblique asymptote for $x\to-\infty$.
Let us analyze the derivative:

$$f'(x)=\frac{x-2}{\sqrt{x^2-4x+3}}=0 \quad\Rightarrow\quad x=2.$$

Since $x=2$ does not belong to the domain, there are no critical points verifying $f'(x)=0$. $x=1$ and $x=3$ are critical points with vertical tangent.

$$\lim_{x\to 1^-}\frac{x-2}{\sqrt{x^2-4x+3}}=-\infty, \qquad \lim_{x\to 3^+}\frac{x-2}{\sqrt{x^2-4x+3}}=\infty,$$

As $f'>0$ on $(3,\infty)$, f is increasing on $(3,\infty)$. As $f'<0$ on $(-\infty,1)$, f is decreasing on $(-\infty,1)$. Since $f(1)=f(3)=0$, these are the global minima, because $f(x)\geq 0$.

Though it is not asked to study concavity, if we calculate the second derivative

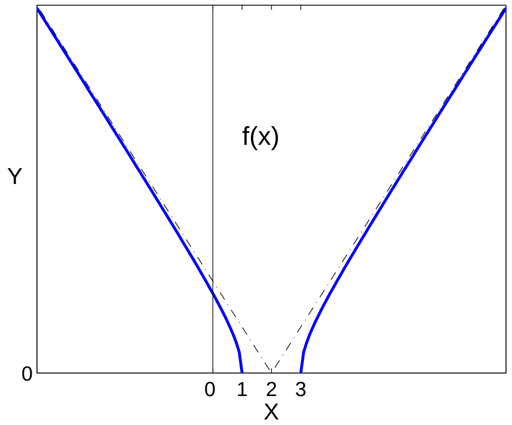
$$f''(x)=\frac{-1}{(x^2-4x+3)^{3/2}},$$

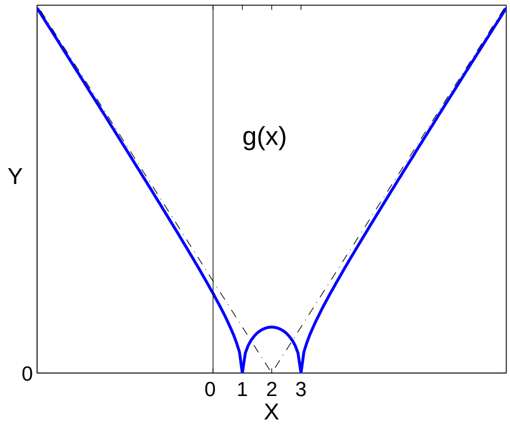
it is always negative, hence f is concave.

b) g is identical to f on the domain of f, but now the domain of g is $\mathbb{R}$.

$$g(x)=\begin{cases} f(x), & x\notin(1,3)\\ \sqrt{-x^2+4x-3}, & x\in(1,3).\end{cases} \quad\Rightarrow\quad g'(x)=\begin{cases} f'(x), & x\notin[1,3]\\ \dfrac{2-x}{\sqrt{-x^2+4x-3}}, & x\in(1,3).\end{cases}$$

$x=2$ is a critical point and a local maximum since g' is positive on $(1,2)$ and negative on $(2,3)$. The points $x=1$ and $x=3$ are still the global minima.
g'' is negative on $(1,3)$, so g is concave on $(-\infty,1)\cup(1,3)\cup(3,\infty)$.





Problem 2

a) The sequence is monotonically decreasing, $b_1 = 2 + \frac{10}{3} < 2 + 4 = 6 < 10 = b_0$ and by induction, if $b_n < b_{n-1}$ then:

$$b_{n+1} = 2 + \frac{b_n}{3} < 2 + \frac{b_{n-1}}{3} = b_n.$$

Since $b_0 > 0$, all b_n are positive, because they are constructed by adding positive numbers, thus the sequence is bounded from below. We conclude that the sequence is convergent.

b) Because of the previous part, the limit exists so the limit must verify

$$l = 2 + \frac{l}{3} \qquad \Rightarrow \qquad l = 3.$$

Problem 3

a) Using the cosine series, convergent on $\mathbb{R}$, we have that:

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \quad \Rightarrow \quad \frac{\cos(t) - 1}{t} = \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n-1}}{(2n)!}$$

$$\Rightarrow \quad \int_0^{x^2} \frac{\cos(t) - 1}{t}\, dt = \left[\sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{2n(2n)!} \right]_0^{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2n(2n)!}.$$

b) We can compute the limit with the previous series:

$$\lim_{x \to 0} \frac{g(x)}{x^4} = -\frac{1}{4}.$$

Or applying twice the L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{g(x)}{x^4} = \lim_{x \to 0} \frac{(\cos(x^2) - 1)2x}{x^2 \cdot 4x^3} = \lim_{x \to 0} \frac{\cos(x^2) - 1}{2x^4} = \lim_{x \to 0} \frac{-\sin(x^2)}{4x^2} = -\frac{1}{4}.$$

Problem 4

a) i) Using the quotient test:

$$\lim_{n\to\infty} \frac{(n+3)^4 n!}{(n+2)^4(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1}\left(\frac{n+3}{n+2}\right)^4 = 0,$$

therefore the series is convergent.

ii) Since is an alternating series we use the Leibniz test. The general term is decreasing and

$$\lim_{n\to\infty} \frac{1}{\log n} = 0,$$

hence the series converges, but it is not absolutely convergent because $\frac{1}{\log n} > \frac{1}{n}$ and the harmonic series diverges.

b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+1}}{2^{2n} n!} = 3\sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^n \frac{1}{n!} = 3\left[\sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n \frac{1}{n!} - 1\right] = 3(e^{-3/4} - 1).$$

Problem 5

a) Integrating by parts:

$$\int \arctan x \, dx = x \arctan x - \int \frac{x\,dx}{1+x^2} = x\arctan x - \frac{1}{2}\log(1+x^2) + C.$$

where $u = \arctan x$, $du = \frac{1}{1+x^2}\,dx$, $dv = dx$, $v = x$.

b) Using the volume's formula and doing partial fraction decomposition we have:

$$\begin{aligned} \text{V} &= \pi \int_0^3 \left(\frac{x}{\sqrt{25-x^2}}\right)^2 dx = \pi \int_0^3 \left(-1 + \frac{5/2}{5-x} + \frac{5/2}{5+x}\right) dx \\ &= \pi \left[-x + \frac{5}{2}\log\left|\frac{5+x}{5-x}\right|\right]_0^3 = \pi\left(-3 + \frac{5}{2}\log 4\right) = \pi\left(-3 + 5\log 2\right). \end{aligned}$$