Universidad Carlos III de Madrid
Calculus II
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Conservative Fields

$\mathbf{F}$ is a **gradient vector field** if $\mathbf{F} = \nabla f$ for some real-valued function f.

In $\mathbb{R}^3 \to \mathbf{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$.

$\mathbf{F} \to$ **conservative vector field**, $f \to$ **potential** of $\mathbf{F}$.

Theorem

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and that $\boldsymbol{\sigma} : [a, b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then

$$\int_{\boldsymbol{\sigma}} \nabla f \cdot d\mathbf{r} = f(\boldsymbol{\sigma}(b)) - f(\boldsymbol{\sigma}(a)).$$

Note. if $\mathbf{F} = -\nabla U$ is a conservative force field:

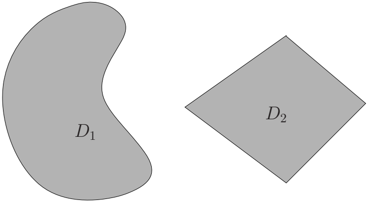
$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla U \cdot d\mathbf{r} = U(\boldsymbol{\sigma}(a)) - U(\boldsymbol{\sigma}(b)).$$

$U \to$ potential energy: gravitational, electric or elastic potential energy.

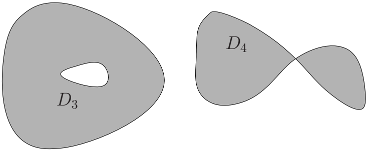
Def.

Let $D \subset \mathbb{R}^n$, D is **simply connected** if every closed curve on D can be contracted continuously to a point.

A simply connected set on $\mathbb{R}^2$ is one whose boundary is formed by a single closed simple curve.



Simply connected sets



Nonsimply connected sets

Theorem

Let $D \subset \mathbb{R}^n$, b simply connected and $\mathbf{F}$ *a* C^1 *vector field defined on* D. *The following are equivalent:*

1. $\mathbf{F}$ *is a conservative vector field on* D, *that is,* $\exists f \in C^2(D)$ *such that* $\mathbf{F} = \nabla f$. *Where* f *is the potential of* $\mathbf{F}$.
2. *For every closed curve* $\boldsymbol{\sigma}$ *on* D,
$$\int_{\boldsymbol{\sigma}} \mathbf{F} \cdot d\mathbf{r} = 0.$$
3. *For every* $\boldsymbol{\sigma}_1$ *and* $\boldsymbol{\sigma}_2$ *curves on* D *with same endpoints, we have*
$$\int_{\boldsymbol{\sigma}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\boldsymbol{\sigma}_2} \mathbf{F} \cdot d\mathbf{r}.$$
4. $n = 2$. *If* $\mathbf{F} = (P, Q)$ *then* $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

   $n = 3$. $\nabla \times \mathbf{F} = 0$.

Remark: If $\mathbf{F} = (F_1, F_2, F_3) \in C^1 \rightarrow$ **curl** of $\mathbf{F} \colon \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is:

$$\operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix},$$

with the property: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0.$