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Unit 3

Integrals Depending on a Parameter

Definition 3.1. *Let* $f\colon [a,b]\times[c,d]\to\mathbb{R}$*, if for each fixed* $t\in[c,d]$ *the function* $f(x,t)$ *is integrable over* $[a,b]$ *on the* x *variable, we define the following function* $F\colon [a,b]\to\mathbb{R}$ *as*

$$F(t)=\int_a^b f(x,t)dx.$$

We call $F(t)$ *an* **integral depending on a parameter.**

Theorem 3.2. f *continuous on* $[a,b]\times[c,d]\Rightarrow F$ *continuous on* $[c,d]$*.*

Theorem 3.3. f *and* $f_t=\dfrac{\partial f}{\partial t}$ *continuous on* $[a,b]\times[c,d]\Rightarrow F$ *differentiable on* $[c,d]$ *and*

$$F'(t)=\int_a^b f_t(x,t)dx.$$

Note. The idea is that we can interchange the processes of integration and differentiation.

Theorem 3.4 (Leibniz's Theorem)**.** *Let* f *and* $f_t=\dfrac{\partial f}{\partial t}$ *be continuous functions on* $[a,b]\times[c,d]$ *and* α,β *differentiable functions on* $[c,d]$ *with image on* $[a,b]$*, that is,* $\alpha(t),\beta(t)\colon [c,d]\to[a,b]$*,* $x\in[\alpha(t),\beta(t)]\subset[a,b]$*. We define*

$$G(t)=\int_{\alpha(t)}^{\beta(t)} f(x,t)dx,$$

then G *is differentiable on* $[c,d]$ *and*

$$G'(t)=f(\beta(t),t)\cdot\beta'(t)-f(\alpha(t),t)\cdot\alpha'(t)+\int_{\alpha(t)}^{\beta(t)} f_t(x,t)dx$$

The previows theorems cannot be used when the integrals are improper. So we have the following results that are valid if the integrals are improper or not.

Theorem 3.5. *Let $f(x,t)$ be a continuous function of t on $[c,d]$ for almost every $x \in [a,b]$, if there exists a function $g(x)$ integrable on $[a,b]$ such that $|f(x,t)| \leq g(x)$, $\forall t \in [c,d]$ and for almost every $x \in [a,b]$, then F* ***is continuous*** *on $[c,d]$.*

Theorem 3.6. *Let $f_t(x,t)$ be a continuous function of t on $[c,d]$ for almost every $x \in [a,b]$, if there exists a function $g(x)$ integrable on $[a,b]$ such that $|f_t(x,t)| \leq g(x)$, $\forall t \in [c,d]$ and for almost every $x \in [a,b]$, then F* ***is differentiable*** *on $[c,d]$ and*

$$F'(t) = \int_a^b f_t(x,t)dx.$$

Theorem 3.7. *Let $f(x,t)$ be integrable over $[a,b] \times [c,d]$, then $F(t)$ is integrable over $[c,d]$ and*

$$\int_c^d F(t)dt = \int_c^d \int_a^b f(x,t)dxdt = \int_a^b \int_c^d f(x,t)dtdx.$$

Note. The integrals that depend on a parameter can be computed by taking the derivative with respect to the parameter, $F'(t) = \int_a^b f_t(x,t)dx$, if we are able to compute this integral, then we just have to integrate with respect to the parameter, and finally to find the value of the constant of integration that appears, fixing the value of the parameter.
We can also use this idea to compute an integral I that does not depend on a parameter, by making it depend on a parameter, $I(t)$, then we compute $I(t)$ by differentiating and integrating with respect to t, and finally fixing the parameter to t_0 that gives us back to the value of $I = I(t_0)$.

The Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

It is a generalization of the factorial function to real and complex numbers.

Properties of the Gamma function

1. $\Gamma(x)$ is continuous and differentiable.
2. $\Gamma(x) \in C^\infty$ and $\frac{d^n}{dx^n}\Gamma(x) = \int_0^\infty (\log t)^n t^{x-1} e^{-t} dt.$
3. $\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(1/2) = \sqrt{\pi}.$
4. $\Gamma(x+1) = x\Gamma(x). \quad \Gamma(n+1) = n!,\ n \in \mathbb{N}.$
5. $\lim_{x \to 0^+} \Gamma(x) = \infty.$

The Beta function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad p,q > 0$$

It is an integral depending on two parameters, all the theory is valid with the natural modifications.

Properties of the Beta function

1. $B(p,q) = B(q,p).$
2. $B(p,q)$ is continuous and differentiable on each variable.
3. $B(p,q) \in C^\infty$, and
$$\frac{\partial^{n+m}}{\partial p^n \partial q^m} B(p,q) = \int_0^1 x^{p-1}(\log x)^n (1-x)^{q-1} (\log(1-x))^m dx,\ p,q > 0.$$
4. $B(p,q) = \frac{q-1}{p+q-1} B(p,q-1), \quad q > 1.$
5. $B(m+1,n+1) = \frac{1}{m+n+1}\binom{m+n}{n}^{-1},\ m,n \in \mathbb{N}.$
6. $B(p,q) = 2\int_0^{\pi/2} (\cos t)^{2p-1} (\sin t)^{2q-1} dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt.$
7. $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$
8. $B(1/2,1/2) = \pi.$

The Transform of Laplace

It is useful to solve some integrals, solve systems of ordinary differential equations with initial value and to study systems whose equations contain derivatives and integrals.

Definition 3.8. *Let $f\colon [0,\infty) \to \mathbb{R}$ be integrable, with exponential growth, that is, $|f(x)| \leq Ce^{\alpha x}, \forall x > T$, where C, α, T are constants depending of f, we define the* **Laplace Transform of** f *as*

$$L[f(x)](s) \equiv F(s) = \int_0^\infty e^{-sx} f(x)dx.$$

Properties of the Transform of Laplace

1. $L[f(x)](s)$ converges for $s \in (\alpha, \infty)$ and is continuous on (α, ∞).
2. $L[f(x)](s) \leq \dfrac{C}{s-\alpha}, \quad s > \alpha.$
3. Linearity: $L[\alpha f(x) + \beta g(x)](s) = \alpha L[f(x)](s) + \beta L[g(x)](s)$.
4. $L[1](s) = \dfrac{1}{s}, \quad s > 0. \qquad L[e^{ax}](s) = \dfrac{1}{s-a}, \quad s > a.$

   $L[x^n](s) = \dfrac{n!}{s^{n+1}}, \quad s > 0,\ n \in \mathbb{N}. \qquad L[x^\alpha](s) = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad s > 0,\ \alpha > -1.$

   $L[\sin{(ax)}](s) = \dfrac{a}{s^2+a^2}, \quad s > 0. \qquad L[\cos{(ax)}](s) = \dfrac{s}{s^2+a^2}, \quad s > 0.$
5. Translation: $L[e^{-ax} f(x)](s) = L[f(x)](s+a) = F(s+a), \quad a \in \mathbb{R}.$
6. Redefining $f(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases} \Rightarrow L[f(x-a)](s) = e^{-as} L[f(x)](s), \quad a > 0.$
7. $L[f(ax)](s) = \dfrac{1}{a} L[f(x)]\left(\dfrac{s}{a}\right), \quad a > 0.$
8. $\dfrac{d^n}{ds^n} L[f(x)](s) = (-1)^n L[x^n f(x)](s), \quad n = 1, 2, 3\ldots$
9. Let $f^{(n-1)}$ be differentiable on $(0,\infty)$ and $f, f', \ldots, f^{(n-1)}$ continuous on $x \geq 0$, then

   $$L\left[\frac{d^n}{dx^n} f(x)\right](s) = s^n L[f(x)](s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots . - f^{(n-1)}(0).$$

   In particular

   $$L[f'(x)](s) = sL[f(x)](s) - f(0),$$

   $$L[f''(x)](s) = s^2 L[f(x)](s) - sf(0) - f'(0).$$

Definition 3.9. *The* **convolution** *of $f(x)$ and $g(x)$ is the function*

$$f \star g(x) = \int_0^x f(u)g(x-u)du, \ x \geq 0.$$

Properties

$f \star g(x) = g \star f(x)$.

$L[f \star g(x)](s) = F(s)G(s)$.

Definition 3.10. *Given a function $F(s)$, if there exists $f(x)$, continuous on $x \geq 0$, such that $L[f(x)](s) = F(s)$, we define the* **Inverse Laplace Transform** *of $F(s)$ as $L^{-1}[F(s)](x) = f(x)$.*

The Inverse Laplace Transform is also linear.

Application of The Transform of Laplace

We can use the Laplace transform to solve systems of differential equations: