Unit 2

Integration in Several Variables

The n-dimensional integral

Here we will give the definition of the n-dimensional integral, and then we will particularize to the special case of $n = 2, 3$.

Definition 2.1. *Let R be a n-dimensional rectangle, $R = [a_1, b_1]\times[a_2, b_2]\times\cdots\times[a_n, b_n]$, with $a_i, b_i \in \mathbb{R},\ i = 1, \ldots, n$. We define the* **measure** *of R as*

$$|R| = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n).$$

Note. For $n = 1$ the measure is just the length of the interval, for $n = 2$, the area, and for $n = 3$, the volume.

Definition 2.2. *A* **partition** *P of the n-dimensional rectangle R is a set of partitions of each interval $[a_i, b_i],\ i = 1, \ldots, n$. Such that $R = \bigcup_{i=1}^{N} R_i$ is a union of subrectangles.*

Definition 2.3. *Let f be a bounded function defined on an n-dimensional rectangle R and P a partition of R such that $R = \bigcup_{i=1}^{N} R_i$, we define the*

$$\begin{aligned}
\textit{Upper sum of } f \textit{for the partition } P \quad &\rightarrow \quad U(f, p) = \sum_{i=1}^{N} M_i |R_i|, \\
\textit{Lower sum of } f \textit{for the partition } P \quad &\rightarrow \quad L(f, p) = \sum_{i=1}^{N} m_i |R_i|, \\
\textit{Riemann sum of } f \textit{for the partition } P \quad &\rightarrow \quad R(f, p) = \sum_{i=1}^{N} f(c_i) |R_i|,
\end{aligned}$$

where $M_i = \sup\{f(x) : x \in R_i\},\ m_i = \inf\{f(x) : x \in R_i\}$, and $c_i \in R_i$ is any point, $i = 1, \ldots, n$.

Theorem 2.4. *Let f be a bounded function on R, then*

$$\sup_P\{L(f, P)\} \leq \inf_P\{U(f, P)\}.$$

Definition 2.5. *Let f be a bounded function on R, if there exists a real number I verifying*

$$I = \sup_P \{L(f,P)\} = \inf_P \{U(f,P)\},$$

we say that f is Riemann integrable over R. The number I is called the $\boxed{\textit{Riemann integral}}$ *of f over R, and we will denote it by*

$$I = \int_R f = \int_R f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n.$$

Note. If f is integrable over R, then I is the unique number verifying $\boxed{L(f,P) \le I \le U(f,P)}$ forall given partition P of R.

Theorem 2.6. *Let f be a bounded function on R, if there is a sequence of partitions $\{P_n\}$ of R such that*

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n),$$

we say that f is integrable over R and that

$$\int_R f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

Theorem 2.7. *f continuous on $R \Rightarrow f$ integrable over R.*

Theorem 2.8. *If f is bounded on R and the set of discontinuities is formed by a finite or countable union of graphs of continuous functions, then f is integrable over R.*

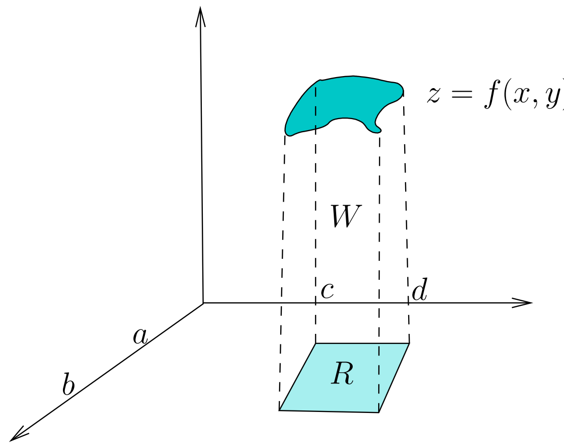
Properties

Let f, g be integrable functions, $c \in \mathbb{R}$. Then

1. $f + g$ is integrable over R and $\displaystyle\int_R f + g = \int_R f + \int_R g$.
2. cf is integrable over R and $\displaystyle\int_R cf = c\int_R f$.
3. $|f|$ is integrable over R and $\displaystyle\left|\int_R f\right| \le \int_R |f|$.
4. If $f \le g$ on R, then $\displaystyle\int_R f \le \int_R g$.
5. If $R_1, \dots, R_m$ are disjoint rectangles such that f is integrable over R_i and $Q = R_1 \cup R_2 \cup \cdots \cup R_m$, then f is integrable over Q and $\displaystyle\int_Q f = \sum_{i=1}^m \int_{R_i} f$.

All the definitions and theorems can be extended to more general regions than n-dimensional rectangles, we will see it in the special case of $n = 2, 3$.

Double Integrals



First of all, let us see the meaning of a double integral. Consider a nonnegative continuous function of two variables $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$, whose domain R is a rectangle with sides parallel to the coordinate axes. R is the cartesian product
$R = [a, b] \times [c, d]$, the graph of $z = f(x, y)$, is a surface lying above the rectangle R. This surface, the rectangle R and the four planes $x = a$, $x = b$, $y = c$ and $y = d$, form the boundary of a region W in the space, the double integral of f over R is the volume of that region W. To compute this volume, we can use Cavalieri's Principle, or volume by parallel cross-sections. If we have a solid body whose cross-sectional area is is given by $A(x)$, for $x \in [a, b]$, according to Cavalieri's Principle, its volume is given by $\displaystyle\int_a^b A(x)\, dx$. Therefore, cutting planes perpendicular to the x plane, the cross-sectional area is $A(x) = \displaystyle\int_c^d f(x, y)\, dy$, obtaining the following formula for the volume

$$V = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.$$

This integral is known as an **iterated integral**, because it is obtained by integrating with respect to y and then by integrating the result with respect to x. If we reverse the roles of x and y, that is, cutting perpendicular planes to the y axis, we obtain that

$$V = \int_c^d \int_a^b A(y)\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy.$$

If the function is not a nonnegative function, that is, it takes positive and negative values, then we shall interpret the integral as a signed volume.

Now we will write **the integral as a limit of a Riemann sum.** Let us consider a function $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ defined on $R = [a, b] \times [c, d]$. Take a regular partition, $\{P_n\}$, of order n: $\Delta x_i = x_{i+1} - x_i = \dfrac{b-a}{n}$ and $\Delta y_j = y_{j+1} - y_j = \dfrac{d-c}{n}$. Thus, we can write the Riemann sum as

$$R(f, P_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(c_{ij})\, \Delta x_i\, \Delta y_j = \sum_{i=1}^{n} \sum_{j=1}^{n} f(c_{ij})\, \Delta A_{ij},$$

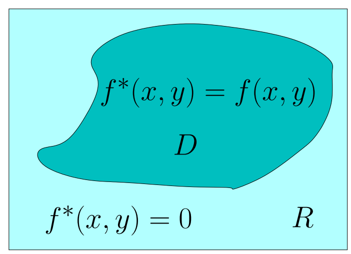
where $\Delta A_{ij} = \Delta x_i\, \Delta y_j$, $c_{ij} \in R_{ij}$, and R_{ij} denotes the ij-th subrectangle of the partition of R. If the function is integrable (that is, it verifies the conditions given before for the n- dimensional case in Definition 2.5, applied to $n = 2$), then we have the Definition of the double integral:

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA := \lim_{n\to\infty} R(f,P_n) = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(c_{ij})\,\Delta A_{ij}.$$

Theorem 2.9. *Fubini's Theorem. Let $f\colon R \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function with a rectangular domain $R = [a,b]\times[c,d]$. Then,*

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The double integral over more general regions



If we wish to define the double integral on a region D more general than a rectangle, first let us see that everything works. We will extend the function $f(x,y)$ defined on D to a rectangle R, verifying $D \subset R$, such that the value of the new function f^* outside D is cero, that is

$$f^*(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \notin D. \end{cases}$$

Therefore, with this extension, the value of the integral is the same as before, and all the definitions, theorems and properties that we have seen, also hold for more general regions:

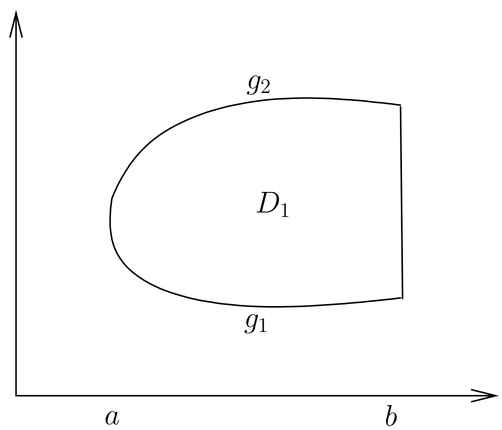
$$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA.$$

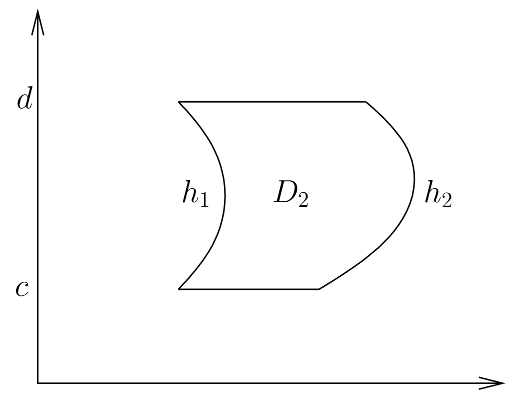
Now we will define three special types of subset of the xy plane, and then we will extend the notion of the double integral to them:

Elementary regions in the plane

We have three types

- Type I: $D_1 = \{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ g_1(x) \le y \le g_2(x)\},\ g_1, g_2$ continuous.
- Type II: $D_2 = \{(x, y) \in \mathbb{R}^2 : y \in [c, d],\ h_1(y) \le x \le h_2(y)\},\ h_1, h_2$ continuous.
- Type III: D_3 : is one that is both type I and type II simultaneously.





Any other region can be considered as the union of these elementary regions. To compute a double integral over these elementary regions we have the following theorem:

Theorem 2.10. *The double integral of an elementary region of*

- *Type I, is:*

$$\iint_{D_1} f(x, y)\, dA = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

- *Type II, is:*

$$\iint_{D_2} f(x, y)\, dA = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx.$$

Changing the order of integration

If D is of type III we can use either both ways to compute the integral, sometimes one of the iterated integrals can be more difficult to compute than the other, so it is very useful to change the order of integration in these cases.

Triple Integrals

As we did before for the double integral, consider a function $f\colon R \subset \mathbb{R}^3 \to \mathbb{R}$, defined on $R = [a,b] \times [c,d] \times [e,l]$, a regular partition, $\{P_n\}$, of order n:
$\Delta x_i = x_{i+1} - x_i = \dfrac{b-a}{n}$, $\Delta y_j = y_{j+1} - y_j = \dfrac{d-c}{n}$, and $\Delta z_k = z_{k+1} - z_k = \dfrac{l-e}{n}$.

Then, we can write the Riemann sum as

$$R(f,P_n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(c_{ijk})\,\Delta x_i\,\Delta y_j\,\Delta z_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(c_{ijk})\,\Delta V_{ijk},$$

where $\Delta V_{ij} = \Delta x_i\,\Delta y_j\,\Delta z_k$, $c_{ijk} \in R_{ijk}$, and R_{ijk} is the ijk-th rectangular box of the partition of R. If the function is integrable then we have the

Definition of the triple integral:

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz = \iiint\limits_R f(x,y,z)\,dV := \lim_{n\to\infty} R(f,P_n) =$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(c_{ijk})\,\Delta V_{ijk}.$$

Theorem 2.11. Fubini's Theorem. *Let $f\colon R \subset \mathbb{R}^3 \to \mathbb{R}$ be an integrable function with a rectangular domain $R = [a,b] \times [c,d] \times [e,l]$. Then,*

$$\begin{aligned}
\iiint\limits_R f(x,y,z)\,dV &= \int_e^l \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_c^d \int_e^l \int_a^b f(x,y,z)\,dx\,dz\,dy \\
&= \int_c^d \int_a^b \int_e^l f(x,y,z)\,dz\,dx\,dy = \\
&= \text{etc. (six posible iterated integrals).}
\end{aligned}$$

Now, we will consider **more general regions** as we did in the two dimensional case. Given $f\colon W \subset \mathbb{R}^3 \to \mathbb{R}$, include the region on a rectangular box $W \subset R$ and extend f to a function f^*, defined on R, that agrees with f on W and is zero outside W. If R is a box containing W, and ∂W consists of graphs of finitely many functions, then f^* will be integrable and

$$\iiint\limits_W f(x,y,z)\,dV = \iiint\limits_R f^*(x,y,z)\,dV.$$

The value of the integral is independent of the choice of R.

Elementary regions in the space

We have four types

- Type I: Top and bottom are surfaces, $z = h(x, y)$.
  $W_1 = \{(x, y, z) \in \mathbb{R}^2 : x \in [a, b],\ g_1(x) \le y \le g_2(x), h_1(x, y) \le z \le h_2(x, y)\}$,
  where g_1, g_2, h_1, h_2 are continuous functions. The integral over W_1 will be:

$$\iiint\limits_{W_1} f(x, y, z)\, dV = \int_a^b dx \int_{g_1(x)}^{g_2(x)} dy \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz.$$

  Also Type I is
  $\widetilde{W}_1 = \{(x, y, z) \in \mathbb{R}^2 : y \in [c, d],\ k_1(y) \le x \le k_2(y), h_1(x, y) \le z \le h_2(x, y)\}$,
  where k_1, k_2, h_1, h_2 are continuous functions. The integral over $\widetilde{W}_1$ will be:

$$\iiint\limits_{\widetilde{W}_1} f(x, y, z)\, dV = \int_c^d dy \int_{k_1(y)}^{k_2(y)} dx \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz.$$

- Type II: Front and rear are surfaces, $x = h(y, z)$. Change the roles of $x \leftrightarrow z$.
- Type III: Left and right are surfaces, $y = h(x, z)$. Change the roles of $y \leftrightarrow z$.
- Type IV: Type I, II and III simultaneously.

Change of variables

Let us remember the formula of the one dimensional case:
$\displaystyle\int_a^b f(x)\, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\underline{g'(t)\, dt}$. Here, $g'(t) \to$ **measures how the length of the interval is changed by the transformation.** $\boxed{g'(t)dt \text{ becomes } |J(u, v)|\, du\, dv}$ in the two dimensional case and $\boxed{|J(u, v, w)|\, du\, dv\, dw}$ in the three dimensional, where J is de Jacobian Determinant, that is a measure of how the change of variables transformation distorts the area and the volume, respectively, of the region.

Definition 2.12. *Let $T\colon D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u, v)$ and $y = y(u, v)$. The* **Jacobian** *of T is the determinant of the derivative matrix* $\mathbf{DT}(\mathbf{x}, \mathbf{y})$ *of T:*

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

Theorem 2.13. *Change of variables for double integrals.* *Let D and D^* be elementary regions in the plane and let $T\colon D^* \to D$ be of class C^1 (that is: $T, \dfrac{\partial T}{\partial u}, \dfrac{\partial T}{\partial v}$ continuous). Suppose that T is one-to-one in the interior of D^*. Furthermore, suppose that $D = T(D^*)$. Then, for any integrable function $f\colon D \to \mathbb{R}$, we have*

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D^*} f(x(u,v), y(u,v)) \cdot |J(u,v)|\, du\, dv.$$

Note. $|J(u,v)|$ is the absolute value of the Jacobian determinant.

Similarly for triple integrals:

Definition 2.14. *Let $T\colon W^* \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 transformation given by $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$. The* **Jacobian** *of T is the determinant of the derivative matrix $\mathbf{DT(x,y,z)}$ of T:*

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Theorem 2.15. *Change of variables for triple integrals.* *Let W and W^* be elementary regions in the space and let $T\colon W^* \to W$ be of class C^1. Suppose that T is one-to-one in the interior of D^*. Furthermore, suppose that $W = T(W^*)$. Then, for any integrable function $f\colon W \to \mathbb{R}$, we have*

$$\iiint\limits_{W} f(x,y,z)dxdydz = \iiint\limits_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot |J(u,v,w)|\, du\, dv\, dw.$$

Note. There is an analogous formula for the n-dimensional case.

Some Changes of variables

Polar coordinates in $\mathbb{R}^2$: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty), \theta \in [0, 2\pi] \rightarrow$

$$J(r,\theta) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ sin\theta & r\cos\theta \end{vmatrix} = r.$$

Notice that is not one-to-one, because $(0,\theta) \rightarrow (0,0)$, but it is one-to-one in the interior of D^*. We use polar coordinates when we have symmetry around the origin.

Cylindrical coordinates in $\mathbb{R}^3$: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $r \in [0,\infty), \theta \in [0, 2\pi], z \in \mathbb{R} \rightarrow$

$$J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

We use cylindrical coordinates when there is symmetry around the z-axis. We can define analogous changes interchanging the roles of $x \leftrightarrow z$ and $y \leftrightarrow z$, when the symmetry is around the x or y axis, respectively.

Spherical coordinates in $\mathbb{R}^3$: $\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases}$, $r \in [0,\infty), \theta \in [0, 2\pi], \phi \in [0,\pi] \rightarrow$

$$J(r,\theta,\phi) = \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} = -r^2\sin\phi.$$

Thus $|J(r,\theta,\phi)| = |r^2\sin\phi|$. Useful when there is spherical symmetry around the origin.

Mean Value Theorems

Theorem 2.16 (Mean Value Theorem for double integrals)**.** *Suppose $f\colon D \rightarrow \mathbb{R}$ is continuous and D is an elementary region in R^2. Then, for some point $(x_0, y_0) \in D$ we have that*

$$\iint_D f(x,y)\,dA = f(x_0,y_0)\,A(D), \text{ where } A(D) \text{ denotes the area of } D.$$

Theorem 2.17 (Mean Value Theorem for triple integrals)**.** *Suppose $f\colon W \rightarrow \mathbb{R}$ is continuous and W is an elementary region in R^3. Then, for some point $(x_0, y_0, z_0) \in W$ we have that*

$$\iiint_W f(x,y,z)\,dV = f(x_0,y_0,z_0)\,Vol(W), \text{ where } Vol(W) \text{ denotes the volume of } W.$$

Applications

Areas and volumes

- $\displaystyle\iint_D f(x,y)\,dA$. Volume under f and over the xy plane if $f(x,y) \geq 0$ on D.
- $\displaystyle A(D) = \iint_D 1\,dA$. Area of D.
- $\displaystyle Vol(W) = \iiint_W 1\,dV$. Volume of W.

Average values

- Discrete set: $\displaystyle [x_i]_{av} = \frac{1}{n}\sum_{i=1}^{n} x_i$.
- 1-dim: $\displaystyle \frac{1}{b-a}\int_a^b f(x)\,dx$. Average of a function defined on $[a,b]$.
- 2-dim: $\displaystyle \frac{1}{A(D)}\iint_D f(x,y)\,dx\,dy$. Average of a function defined on D.
- 3-dim: $\displaystyle \frac{1}{Vol(W)}\iiint_W f(x,y,z)\,dx\,dy\,dz$. Average of a function defined on W.

Centers of mass

- Discrete set: $\displaystyle \mathbf{r}_{CM} = (x_{CM}, y_{CM}, z_{CM}) = \frac{\sum_{i=1}^{n} M_i \mathbf{r_i}}{M}$. Where M is the total mass of the system.
- 1-dim: $\displaystyle x_{CM} = \frac{1}{M}\int_a^b x\rho(x)\,dx$. Linear body with mass density $\rho(x)$ defined on $[a,b]$. $M = \int_a^b \rho(x)\,dx$ is the mass of the body.
- 2-dim: $\displaystyle (x_{CM}, y_{CM}) = \frac{1}{M}\left(\iint_D x\rho(x,y)\,dx\,dy, \iint_D y\rho(x,y)\,dx\,dy\right)$.
  Plane body with mass density $\rho(x,y)$ defined on D.
  $M = \iint_D \rho(x,y)\,dx\,dy$ is the mass of the body.

- 3-dim:
  $x_{CM} = \dfrac{1}{M} \displaystyle\iiint_W x\rho(x,y,z)\,dx\,dy\,dz,$
  $y_{CM} = \dfrac{1}{M} \iiint_W y\rho(x,y,z)\,dx\,dy\,dz,$
  $z_{CM} = \dfrac{1}{M} \iiint_W z\rho(x,y,z)\,dx\,dy\,dz,$

  Three-dimensional body with mass density $\rho(x,y,z)$ defined on W.
  $M = \iiint_W \rho(x,y,z)\,dx\,dy\,dz$ is the mass of the body.

Moments of inertia of a rigid homogeneous body about an axis

The moment of inertia of a body about an axis measures its response to efforts to rotate around the axis.

- Discrete mass m at a distance d of an axis: $I_{axis} = md^2$.
- 2-dim:
  $I_{axis} = \displaystyle\iint_D \rho(x,y)d^2(x,y)\,dx\,dy$, if the distance to the axis is given by $d(x,y)$.

  $I_x = \displaystyle\iint_D \rho(x,y)y^2\,dx\,dy,\ I_y = \iint_D \rho(x,y)x^2\,dx\,dy.$
- 3dim: $I_{axis} = \displaystyle\iiint_W \rho(x,y,z)d^2(x,y,z)\,dx\,dy\,dz$, if the distance to the axis is given by $d(x,y,z)$.

  $I_x = \displaystyle\iiint_W \rho(x,y,z)(y^2+z^2)\,dx\,dy\,dz,\ I_y = \iiint_W \rho(x,y,z)(x^2+z^2)\,dx\,dy\,dz,$

  $I_z = \displaystyle\iiint_W \rho(x,y,z)(x^2+y^2)\,dx\,dy\,dz.$