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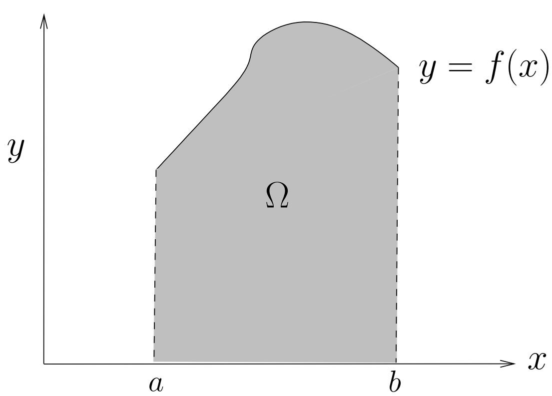
Unit 1

Integration in One Variable

The Riemann Integral

Geometrically, we use the derivative of a function to get the slope of the function at a given point. The integral of a function gives us the area under the curve.

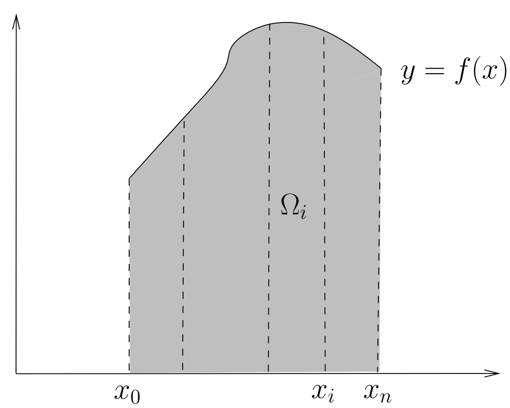


The integral represents the **area under the graph** of the function $f(x) \geq 0$ and over the $x-$axis on the interval $x \in [a, b]$.

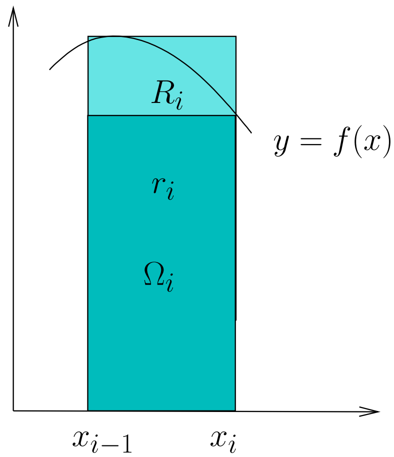
Dividing the whole interval into n subintervals $[x_0, x_1]$,$[x_1, x_2]$,. . ., $[x_{n-1}, x_n]$, where $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. We have now n vertical strips, the total area is just the sum of the areas of each individual vertical strip. We define

$$M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

as the maximum value that $f(x)$ takes on the interval $[x_{i-1}, x_i]$. Similarly,



$$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$



Now, let us consider the rectangles r_i and R_i such that

$$r_i \leq \Omega_i \leq R_i,$$

then, we have that

$$\text{area}(r_i) \leq \text{area}(\Omega_i) \leq \text{area}(R_i),$$
$$m_i(x_i - x_{i-1}) \leq \text{area}(\Omega_i) \leq M_i(x_i - x_{i-1}).$$

Summing for all the intervals, we obtain that

$$m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) \leq \text{area}(\Omega) \leq$$
$$\leq M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

Let us now establish more precisely all these concepts:

Definition 1.1. *A partition of the interval* $[a, b]$ *is a finite sequence of numbers* $P = \{x_0, x_1, \cdots, x_n\}$ *such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We can think that the points $x_0, x_1, \cdots, x_n$ divide the interval $[a, b]$ into the subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$. The number n of intervals can be as small as 1 or as large as we wish.

Definition 1.2. *Let* f *be a bounded function defined on* $[a, b]$ *and* $P = \{x_0, x_1, \cdots, x_n\}$ *a partition of* $[a, b]$*, define*

$$\begin{aligned} m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\}, \\ M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\}. \end{aligned}$$

We call the **Lower sum** *of* f *for* P *on* $[a, b]$*, the number*

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and the **Upper sum** *of* f *for* P *on* $[a, b]$*, the number*

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i \Delta x_i.$$

We observe from the definition that for **any partition** P we have the property

$$L(f, P) \leq U(f, P).$$

Definition 1.3. *Let P and P' be any two partitions of $[a, b]$, we say that P' is a* **refinement of** *P or that is* **finer than** *P if every point of P is contained in P', that is, if every subinterval of P is contained in P'.*

Lemma 1.4. *If P' is finer than P, then we can assert that*

$$\begin{aligned} L(f, P) &\leq L(f, P'), \\ U(f, P') &\leq U(f, P). \end{aligned}$$

The case $L(f, P) \leq L(f, P')$ follows from the fact that the minimum of f on an interval is less than or equal to the minimum on any interval contained in it. We can proof the second case in a similar way.

Theorem 1.5. *If f is a bounded function on $[a, b]$ and P_1, P_2, are two any partitions of $[a, b]$. Then,*

$$L(f, P_1) \leq U(f, P_2).$$

Proof. To prove this, let Q be a partition of the interval that refines both P_1 and P_2, which we can arrange by using all the subdivision points of both of them $Q = P_1 \cup P_2$. Then,

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2).$$

□

Definition 1.6. *Let f be a bounded function on $[a, b]$, we say that f is Riemann integrable on $[a, b]$ if*

$$\sup \{L(f, P),\ P \text{ partition of } [a, b]\} = \inf \{L(f, P),\ P \text{ partition of } [a, b]\}.$$

In this case, we define the **Riemann Integral** *as the common value, and we will denote it by*

$$\int_a^b f(x)\,dx = \int_a^b f.$$

We will call it also the **Definite Integral.**

From now on, we will omit the word "Riemann", and we will just use the words integrable and integral. The function which is being integrated is called the integrand, and the numbers a and b are called limits of integration.

If f is integrable, we have the following **property** for any partition P of the interval:

$$L(f, P) \leq \int_a^b f \leq U(f, P).$$

Theorem 1.7. *[Characterization of Riemann integrable functions] Let f be a bounded function on $[a, b]$. f is integrable if and only if $\forall \epsilon > 0$ there is a partition P_ϵ of $[a, b]$ such that* $0 \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$

Theorem 1.8. *Let f be a continuous function on $[a, b]$, a finite interval, then f is integrable on $[a, b]$.*

Proof. Continuity means that $\forall \epsilon > 0$ there exists a $\delta > 0$ such that $\forall x, y \in [a, b]$ verifying that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. We can take partitions P such that $|x_i - x_{i-1}| < \delta$, then, by Theorem (1.7),

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) = \epsilon(b - a) = \epsilon',$$

verifies the Riemann condition, thus, it is integrable. □

We can generalize the previous theorem:

Theorem 1.9. *Let f be a bounded function with a finite or countable number of points of discontinuity on $[a, b]$, a bounded interval, then f is integrable.*

continuous $\Rightarrow$ integrable
integrable $\not\Rightarrow$ continuous

We can generalize the definition of Lower and Upper sums to the so called Riemann sums, where we approximate the value of the function on each subinterval by any of the values taken by f on it:

Definition 1.10. *Let f be bounded on $[a, b]$ and $P = \{x_i\}_{i=0}^{n}$ a partition of the interval. We call the* Riemann sum *of f for the partition P, the value*

$$R(f, P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i)\Delta x_i, \text{ where } c_i \in [x_{i-1}, x_i] \text{ is any point.}$$

Note. The notation $\int_a^b f(x)\,dx$ for the integral is due to **Leibniz.** Where the symbol "$\int$", called an integral sign, is an elongated S which replaces the Greek symbol "Σ" of ordinary summation. Similarly, the "dx" replaces the "Δx_i" of the summation formula:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i \to \int_a^b f(x)\,dx$$

Theorem 1.11. *Let f be continuous on $[a, b]$. Then, for any $\epsilon > 0$ there exists $\delta > 0$, such that for any partition $P = \{x_i\}_{i=0}^{n}$ of $[a, b]$ verifying that $|x_i - x_{i-1}| < \delta$, for $c_i \in [x_{i-1}, x_i]$ and for any Riemann sum, we have that*

$$\left| R(f, P) - \int_a^b f \right| < \epsilon.$$

Proof. As f is continuous, it is integrable, so $U(f, P) - L(f, P) < \epsilon$.
Since $L(f, P) \leq \displaystyle\int_a^b f \leq U(f, P)$ and $L(f, P) \leq R(f, P) \leq U(f, P)$, we have that

$$\left| R(f, P) - \int_a^b f \right| < |U(f, P) - L(f, P)| = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \epsilon'(b - a) = \epsilon^*.$$

□

The last theorem says that we can approximate the value of the integral of a continuous function as much as we want by a Riemann sum, just by making the partition small enough.

Let us establish now some basic properties of the integral:

Properties of the Integral

1. $\int_a^b c_1 f + c_2 g = c_1 \int_a^b f + c_2 \int_a^b g.$
2. $\int_a^b f = \int_a^c f + \int_c^b f.$
3. $\int_a^b f = -\int_b^a f.$
4. $\int_a^a f = 0.$
5. $\int_a^b fg \neq \int_a^b f \int_a^b g.$
6. $f \geq g \Rightarrow \int_a^b f \geq \int_a^b g.$
7. $f \geq 0 \Rightarrow \int_a^b f \geq 0,$

   If $f \leq 0 \Rightarrow \int_a^b f \leq 0.$
8. $\left| \int_a^b f \right| \leq \int_a^b |f|.$
9. $m \leq f(x) \leq M, \ \forall x \in [a,b] \Rightarrow$

   $m(b-a) \leq \int_a^b f(x) \leq M(b-a).$

Theorem 1.12 (First Mean Value Theorem for integrals)**.** *Let f be continuous on $[a,b]$. Then, $\exists\, x_0 \in [a,b]$ such that*

$$\int_a^b f = f(x_0)(b-a).$$

$\frac{1}{b-a}\int_a^b f$ *is called the average of f over $[a,b]$.*

Theorem 1.13 (Second Mean Value Theorem for integrals)**.** *Let f be continuous on $[a,b]$ and g integrable such that g does not change sign on $[a,b]$. Then, $\exists\, x_0 \in [a,b]$ such that*

$$\int_a^b fg = f(x_0)\int_a^b g.$$

The Indefinite Integral

Geometrically the problem of differentiation arises when we want to find the slope of a curve and the problem of integration when computing the area under a curve, but Newton found out that differentiation and integration are inverse processes:

- **Differentiation:** Given a function $F(x)$, find a function $f(x)$ satisfying

$$\frac{dF(x)}{dx} = f(x).$$

- **Integration:** Given a function $f(x)$, find a function $F(x)$ satisfying

$$\frac{dF(x)}{dx} = f(x).$$

A function $F(x)$ solving the second problem is called an **antiderivative**, **primitive** or an **indefinite integral** of $f(x)$.
The problem of differentiation has always solution but the problem of integration does not always have a solution and, in general, is more complicated.

Definition 1.14. *Let f be integrable on [a,b], then*

$$F(x) = \int_a^x f(t)\,dt,$$

is an antiderivative of $f(x)$ defined on $[a,b]$.

Theorem 1.15. *f integrable on $[a,b] \Rightarrow F$ continuous on $[a,b]$.*

Proof. If f is bounded on $[a,b]$, then $|f| \leq M$ on $[a,b]$. Thus,

$$|F(x) - F(y)| = \left|\int_a^x f(t)\,dt + \int_y^a f(t)\,dt\right| = \left|\int_y^x f(t)\,dt\right| \leq \left|\int_y^x M\,dt\right| = |M(x-y)| = \\ = M|x-y|.$$

Now, $\forall x, y \in [a,b]$ and $\forall \epsilon \geq 0$, $\exists \delta \geq 0$ such that $|x-y| < \delta$. Therefore,
$|F(x) - F(y)| = M|x-y| < M\delta = \epsilon$, thus F is continuous on $[a,b]$. □

Theorem 1.16. *The Fundamental Theorem of Calculus, FTC*

Let f be integrable on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ defined $\forall x \in [a,b]$.
If f is continuous at $c \in [a,b] \Rightarrow F$ is differentiable at c and $F'(c) = f(c)$.
If f is continuous $\forall x \in [a,b] \Rightarrow F$ is differentiable $\forall x \in [a,b]$ and $F'(x) = f(x)$.

Proof. If f is continuous at $c \in [a,b]$, then it attains its maximum and minimum on $[c, c+h]$, so we have the following inequalities:

$$m_h \leq f \leq M_h \qquad \Rightarrow$$

$$\textstyle\int_c^{c+h} m_h \leq \int_c^{c+h} f \leq \int_c^{c+h} M_h \quad \Rightarrow$$

$$\frac{\int_c^{c+h} m_h}{h} \leq \frac{\int_c^{c+h} f}{h} \leq \frac{\int_c^{c+h} M_h}{h} \quad \Rightarrow$$

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h.$$

Now, taking the limit $h \to 0$ and comparing the first and last equations, we get

$$\lim_{h\to 0} \frac{F(c+h) - F(c)}{h} = F'(c) = f(c) = \lim_{h\to 0} m_h = \lim_{h\to 0} M_h.$$

Theorem 1.17 (Barrow's Rule)**.** *Let f and g be continuous on $[a,b]$ and g differentiable on (a,b), such that $g'(x) = f(x)$, $\forall x \in (a,b)$. Then,*

$$\int_a^b f = \int_a^b g' = g(b) - g(a).$$

Proof.

$$F(x) = \int_a^x f \Rightarrow F' = g' \Rightarrow F = g + C, \text{ then,}$$

$$\int_a^a f = F(a) = g(0) + C \Rightarrow C = -g(a).$$

$$F(x) = g(x) - g(a) \Rightarrow F(b) = g(b) - g(a), \text{ therefore,}$$

$$\int_a^b f = F(b) = g(b) - g(a).$$

□

Remark. A function can be integrable without being the derivative of a function.

Theorem 1.18. *Let f be integrable on $[a,b]$ and $f = g'$ for some function g, then*

$$\int_a^b f = \int_a^b g' = g(b) - g(a).$$

Theorem 1.19 (FTC generalized)**.** *Let* $F(x) = \int_a^x f$, *with* f *integrable,*

- *Let* $H(x) = F(g(x)) = \int_a^{g(x)} f$, *then if* g *is differentiable, we have that*

$$H'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

- *Let* $H(x) = \int_{l(x)}^{g(x)} f$, *then if* g *and* l *are differentiable, we have that*

$$H'(x) = f(g(x))g'(x) - f(l(x))l'(x).$$

Integration Techniques

Let us see here the integration techniques for definite and indefinite integrals, we will start with the basic antiderivatives:

Basic Antiderivatives

$$\int x^n = \frac{x^{n+1}}{n+1} + c, \ n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + c$$

$$\int e^{ax} = \frac{1}{a}e^{ax} + c$$

$$\int \sin x = -\cos x + c$$

$$\int \cos x = \sin x + c$$

$$\int \frac{1}{\cos^2 x} = \tan x + c$$

$$\int \frac{1}{\sin^2 x} = -\cot x + c$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \sinh x = \cosh x + c$$

$$\int \cosh x = \sinh x + c$$

Integration by change of variables (CV)

Let g be defined on $[a, b]$ such that g' is continuous on $[a, b]$, let f be integrable on $[g(a), g(b)]$. Then,

- Definite integral: $\displaystyle\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(t))g'(t)\,dt$
- Indefinite integral: $\displaystyle\int f(x)\,dx = \int f(g(t))g'(t)\,dt$ $\quad \rightarrow$ undo the change

Integration by parts (IBP): $\int u dv = uv - \int v\, du$

Let f and g be defined on $[a, b]$ such that f' and g' are continuous on $[a, b]$. Then,

- Definite integral: $\int_a^b f g' = f g\Big|_a^b - \int_a^b f' g$
- Indefinite integral: $\int f g' = f g - \int f' g$

Rational functions: Partial Fraction Decomposition

$$\int \frac{P(x)}{Q(x)}\, dx \;\rightarrow\; P,\ Q \text{ polynomials}$$

- If $deg(P) \geq deg(Q) \rightarrow$ **divide the polynomials:**
  $P(x) = Q(x)C(x) + R(x) \rightarrow$
  $$\int \frac{P(x)}{Q(x)}\, dx = \int C(x) + \int \frac{R(x)}{Q(x)}\, dx.$$
- $\int \frac{R(x)}{Q(x)}\, dx$ with $deg(R(x)) < deg(Q(x))$:
  i) First, check that the integral is not immediate:
  ln type $\rightarrow \int \frac{2x+3}{x^2+3x+8}\, dx = \ln|x^2+3x+8| + c.$
  arctan type $\rightarrow \int \frac{dx}{x^2+8} = \frac{1}{\sqrt{8}} \arctan \frac{x}{\sqrt{8}} + c.$
  ii) If not $\rightarrow$ **Do partial fraction decomposition.**

Factor in denominator	Term in partial fraction decomposition
$x-b$	$\frac{A}{x-b}$
$(x-b)^k$	$\frac{A_1}{x-b} + \frac{A_2}{(x-b)^2} + \cdots \frac{A_k}{(x-b)^k}, \quad k = 1, 2, 3, \cdots$
$(x-a)^2+b^2$	$\frac{Ax+B}{(x-a)^2+b^2}$
$\left((x-a)^2+b^2\right)^k$	$\frac{A_1x+B_1}{(x-a)^2+b^2} + \cdots + \frac{A_kx+B_k}{\left((x-a)^2+b^2\right)^k}, \quad k = 1, 2, 3, \cdots$

For each factor in the denominator add the corresponding term of the table and compute the unknowns $(A,\ B,\ A_1,\ B_1,\ A_2,\ B_2,\cdots)$ by setting equal denominators. After, compute the integrals of each term.

Irrational functions or integrals involving roots

Do a change of variables that eliminates the roots.

$$\int R\left[\left(\frac{ax+b}{cx+d}\right)^{p_1/q_1},\cdots,\left(\frac{ax+b}{cx+d}\right)^{p_r/q_r}\right]\rightarrow t^m=\frac{ax+b}{cx+d},\ m=lcm(q_1,\cdots,q_r).$$

$R=\dfrac{P}{Q}$ is a rational function of its variables, $P,\ Q$ are polynomials.

$lcm\rightarrow$ least common multiple.

Integrals involving trigonometric functions

- $\int\sin^{2n}x,\ \int\cos^{2n}x\rightarrow$ double angle formulas: $\cos 2x=\cos^2 x-\sin^2 x$.
- $\int\sin^{2n+1}x=\int\sin^{2n}x\sin x=\int(1-\cos^2 x)^n\sin x$.
- $\int\cos^{2n+1}x=\int\cos^{2n}x\cos x=\int(1-\sin^2 x)^n\cos x$.
- $\int\sin mx\cos nx\rightarrow$ trigonometric formulas.
- $\displaystyle\int R(\sin x,\cos x)\rightarrow$

R odd in $\sin x\rightarrow t=\cos x$
R odd in $\cos x\rightarrow t=\sin x$
R even in $\cos x$ and $\sin x\rightarrow t=\tan x$
Rest of problems $\rightarrow t=\tan\left(\dfrac{x}{2}\right),\ \left[\sin x=\dfrac{2t}{1+t^2},\ \ \cos x=\dfrac{1-t^2}{1+t^2},\ \ dx=\dfrac{2dt}{1+t^2}\right]$

Some change of variables

1. $\displaystyle\int R(x,\sqrt{x^2+a^2})\rightarrow x=a\tan t$
2. $\displaystyle\int R(x,\sqrt{x^2-a^2})\rightarrow x=\frac{a}{\cos t}$
3. $\displaystyle\int R(x,\sqrt{a^2-x^2})\rightarrow x=a\sin t$

Applications of the Integral: Areas, Volumes and Length

Areas

- Area between the **graph of a function**, the x-axis, between a and b:
$$A = \int_a^b |f|\, dx.$$

- Area between the **graphs of two functions** f, g, between a and b:
$$A = \int_a^b |f - g|\, dx.$$

- Area using **parametric equations:**
The area between the graph of $x = x(t),\ y = y(t)$ and the x-axis between $t = t_0$ and $t = t_1$ is:
$$A = \left| \int_{t_0}^{t_1} y(t)x'(t)\, dt \right|.$$

- Area using **polar coordinates:**
The area of the graph of $r = r(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is
$$A = \int_\alpha^\beta \frac{1}{2} r^2(\theta)\, d\theta.$$

Volumes

- **Volume by parallel cross-sections:** if $A(x)$ is the area of parallel cross-sections over the entire length of a solid, the volume between $x = a$ and $x = b$ is
$$V = \int_a^b A(x)\, dx.$$

- **The Disk method:** the volume of a solid of revolution obtained by rotating $|f(x)|$ about the x-axis between $x = a$ and $x = b$ is
$$V = \int_a^b \pi (f(x))^2\, dx.$$

- **The Shell method:** the volume of a solid of revolution obtained by rotating $f(x) \geq 0$, $x \in [a, b]$, $a \geq 0$, about the y-axis is
$$V = 2\pi \int_a^b x f(x)\, dx.$$

Lengths

- The **length of an arc of a curve** $f(x)$ between $x = a$ and $x = b$ is

$$L(f) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

- If the curve is given in **parametric form**, the length is

$$L = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Other application: Computation of Limits

With the concept of an integrable function we can compute some type of limits, let us see the following theorem and how can we apply it for our purpose.

Theorem 1.20. *If f is a bounded function on $[a, b]$ and there is a sequence of partitions $\{P_n\}$ of $[a, b]$ such that*

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

Then, f is integrable on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} R(f, P_n).$$

Let us see how can we use this theorem to compute, for instance, the following limit:

$$\lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right].$$

Let us consider the Riemann Sum using the right point for the partition of the interval $[0, 1]$ $P_n = \{i/n\}_{i=1}^n$ where $\delta x = 1/n$, we have the identity

$$\int_0^1 f = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right).$$

The idea is to identify the sum as the lower, upper or Riemann sum of an integral and then, to compute the integral. For the given example, reads as

$$\lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n f\left(\frac{i}{n}\right) =$$

$$\lim_{n\to\infty} \left[\frac{1}{1+(1/n)^2} + \frac{1}{1+(2/n)^2} + \cdots + \frac{1}{1+(1/n)^2}\right] \to f(x) = \frac{1}{1+x^2} \Rightarrow$$

$$\lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right] = \int_0^1 f = \int_0^1 \frac{1}{1+x^2} = \arctan x\Big|_0^1 = \frac{\pi}{4}.$$

Improper Integrals

An improper integral is one for which the integrand goes to $\pm\infty$ between the limits of integration, or which has ∞ and/or $-\infty$ as a limit of integration. That is,

f is not bounded or the interval is not bounded:

$$\int_a^{\infty} f(x)\,dx,$$
$$\int_{-\infty}^{b} f(x)\,dx,$$
$$\int_a^{b} f(x)\,dx, \quad |\lim_{x\to x_0} f(x)| = \infty, \text{ for some } x_0 \in [a,b].$$

Improper integrals are evaluated using limits. If the limit exists and is finite, we say that the integral **converges**, otherwise we say that **diverges**.

Definition 1.21. *Let f be a function defined on an infinite interval and locally integrable (that is, it is integrable on each finite subinterval), we define the* **improper integral** *of f as:*

$$\begin{aligned}
\int_a^{\infty} f(x) &= \lim_{N\to\infty}\int_a^{N} f(x), \text{ if the limit is finite,}\\
\int_{-\infty}^{b} f(x) &= \lim_{N\to\infty}\int_{-N}^{b} f(x), \text{ if the limit is finite,}\\
\int_{-\infty}^{\infty} f(x) &= \int_{-\infty}^{a} f(x) + \int_a^{\infty} f(x), \text{ if both are finite.}
\end{aligned}$$

From the definition, we see that an improper integral is a limit of integrals, so it is a double limit, because an integral itself is a limit.
We cannot compute the last type as the following limit: $\int_{-\infty}^{\infty} f(x) \neq \lim_{N\to\infty}\int_{-N}^{N} f(x)$, these equality only holds if the improper integral converges. If it diverges, it is not true, let us see it in the following example

$$\lim_{N\to\infty}\int_{-N}^{N} x = \lim_{N\to\infty}\left[\frac{x^2}{2}\right]_{-N}^{N} = 0, \text{ but}$$
$$\int_{-\infty}^{\infty} x = \lim_{N\to\infty}\left[\frac{x^2}{2}\right]_{-N}^{0} + \lim_{N\to\infty}\left[\frac{x^2}{2}\right]_{0}^{N} \to \text{ both limits diverge.}$$

Definition 1.22. *Let f be a function verifying $\lim_{x\to\alpha}|f(x)| = \infty$ for some $\alpha \in (a,b)$, a finite interval, such that f is integrable on each closed subinterval of $[a,b]$ that does not contain α. We define the improper integral of f as*

$$\begin{aligned} \int_\alpha^b f(x) &= \lim_{\epsilon\to 0^+}\int_{\alpha+\epsilon}^b f(x), \text{ if the limit is finite,} \\ \int_a^\alpha f(x) &= \lim_{\epsilon\to 0^+}\int_a^{\alpha-\epsilon} f(x), \text{ if the limit is finite,} \\ \int_a^b f(x) &= \int_\alpha^b f(x) + \int_a^\alpha f(x), \text{ if both are finite.} \end{aligned}$$

If any of these quantities is infinite or does not have a limit, we say that the integral diverges.

Again, we cannot compute the last type as a single limit, we must compute both limits.

Definition 1.23. *Consider $\int f$ as one of the three types of improper integrals that we have seen. The improper integral $\int f$ is said* **absolutely convergent (A. C.)** *if $\int|f|$ converges.*
If the improper integral $\int|f|$ diverges but $\int f$ converges, then it is said **conditionally convergent (C. C.)**

Note. A. C. $\Rightarrow$ Convergent
Sometimes it is not easy to see if an improper integral converges or diverges, but we can compare with integrals whose behavior is known. Let us see some theorems to make the comparisons, we will work only the first type of improper integrals ($\int_a^\infty f$):

Theorem 1.24. *If $\int_a^b f$ exists $\forall b \geq a$ and $|f(x)| \leq g(x),\ \forall x \geq M$, for certain $M \in \mathbb{R}$. Then, we have the following statements:*

$$\begin{aligned} \int_a^\infty g \text{ conv.} \quad &\Rightarrow \quad \int_a^\infty f \text{ conv. (A. C.), and } \int_a^\infty f \leq \int_a^\infty g, \\ \int_a^\infty f \text{ div.} \quad &\Rightarrow \quad \int_a^\infty g \text{ div.} \end{aligned}$$

Theorem 1.25. *Let f, g be locally integrable functions such that $\lim_{x\to\infty}\dfrac{f(x)}{g(x)} = l$, and $f(x), g(x) \geq 0$, for $x \geq M$, for certain $M \in \mathbb{R}$. Then, we have the following statements:*

1. *If $l = constant \in (0,\infty)$, then* $\displaystyle\int_a^\infty g \text{ conv.} \iff \int_a^\infty f \text{ conv.}$
2. *If $l = 0$, then* $\displaystyle\int_a^\infty g \text{ conv.} \Rightarrow \int_a^\infty f \text{ conv.}$
3. *If $l = \infty$, then* $\displaystyle\int_a^\infty g \text{ div.} \Rightarrow \int_a^\infty f \text{ div.}$

Remark. The two previous theorems are also valid for the other types of improper integrals, (when $x \to -\infty$ or $x \to \alpha \in \mathbb{R}$), modifying the hypotheses appropriately.

To finish, let us see the following theorem connecting integrals with series:

Theorem 1.26 (Integral test for series)**.** *Consider $f \geq 0$ a monotone decreasing function defined for $x \geq 1$. Let $a_n = f(n)$, then*

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx,$$

have the same behavior, or both converge or both diverge.