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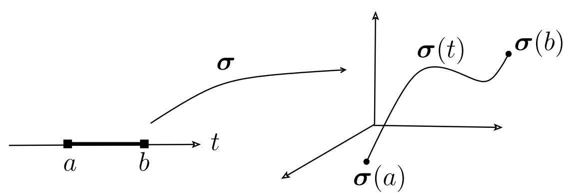
Unit 4

Line and Path Integrals

The Path Integral

Definition 4.1. *A* **path** *or a* **curve** *C is the image of a map $\boldsymbol{\sigma}\colon [a,b] \to \mathbb{R}^n$, $\boldsymbol{\sigma}(t) = (x_1(t), x_2(t)), x_3(t))$. $\boldsymbol{\sigma}$ is called a parametrization of C.*



Definition 4.2. *Consider the path $\boldsymbol{\sigma}\colon [a,b] \to \mathbb{R}^3$, where $\boldsymbol{\sigma}$ is of class C^1, and the composite function $f \circ \boldsymbol{\sigma}(t) = f(x(t), y(t), z(t))$, such that $f \circ \boldsymbol{\sigma}(t)$ is continuous on $[a,b]$.* **The path integral** *or the integral of f along $\boldsymbol{\sigma}$ is*

$$\int_{\boldsymbol{\sigma}} f dr = \int_a^b f(\boldsymbol{\sigma}(t)) \|\boldsymbol{\sigma}'(t)\| dt = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

If $\boldsymbol{\sigma}(t)$ is only piecewise C^1 or $f(\boldsymbol{\sigma}(t))$ is piecewise continuous, we define the path integral by breaking $[a,b]$ into pieces over which $f(\boldsymbol{\sigma}(t))\|\boldsymbol{\sigma}'(t)\|$ is continuous, and summing the integrals over the pieces.
We give the definitions in $\mathbb{R}^3$ but they are valid in R^n by making the natural changes.
Note. $L(\boldsymbol{\sigma}) = \int_{\boldsymbol{\sigma}} 1 dr$ gives us the length of the curve.

The Line Integral

Now we will consider vector fields, that is, functions $\mathbf{F}\colon \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{n}}$, that is, at each point we have a vector.

Definition 4.3. *Let* $\mathbf{F}$ *be a vector field on* R^3 *which is continuous on the* C^1 *path* $\boldsymbol{\sigma}\colon [a,b] \to \mathbb{R}^3$*. We define the* **line integral** *of* $\mathbf{F}$ *along* $\boldsymbol{\sigma}$ *as*

$$\int_{\boldsymbol{\sigma}} \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\boldsymbol{\sigma}(t))\cdot \boldsymbol{\sigma}'(t)dt = \int_a^b \mathbf{F}(x(t),y(t),z(t))\cdot (x'(t),y'(t),z'(t))dt.$$

Notation: $\mathbf{F} = (F_1, F_2, F_3) \to \int_{\boldsymbol{\sigma}} \mathbf{F}\cdot d\mathbf{r} = \int_{\boldsymbol{\sigma}} F_1 dx + F_2 dy + F_3 dz.$

Note. If $\mathbf{F}$ is a force field (electric, gravitational,...) $W = \int_{\boldsymbol{\sigma}} \mathbf{F}\cdot d\mathbf{r}$ $\to$ is the work done by the force field $\mathbf{F}$ on a particle moving along a path $\boldsymbol{\sigma}\colon [a,b] \to \mathbb{R}^3$.

Definition 4.4. *Let* $h\colon I = [a,b] \to I_1 = [a_1,b_1]$ *be a* C^1 *real-valued function that is a one-to-one map of an interval* I *onto another interval* I_1*. Let* $\boldsymbol{\sigma}\colon I_1 \to \mathbb{R}^3$ *be a* C^1 *path. Then we call the composition* $\boldsymbol{\rho} = \boldsymbol{\sigma}\circ h\colon I \to \mathbb{R}^3$ *a* **reparametrization** *of* $\boldsymbol{\sigma}$ *and has the same image in* $\mathbb{R}^3$ *as* $\boldsymbol{\sigma}$*. The reparametrization is*

- *orientation preserving: if* $\boldsymbol{\rho}(a) = \boldsymbol{\sigma}(a)$ *and* $\boldsymbol{\rho}(b) = \boldsymbol{\sigma}(b)$*.*
- *orientation reversing: if* $\boldsymbol{\rho}(a) = \boldsymbol{\sigma}(b)$ *and* $\boldsymbol{\rho}(b) = \boldsymbol{\sigma}(a)$*.*

Theorem 4.5. *Let* $\boldsymbol{\sigma}$ *be a piecewise* c^1 *and* f *a continuous function on the image of* $\boldsymbol{\sigma}$*, and let* $\boldsymbol{\rho}$ *be any reparametrization of* $\boldsymbol{\sigma}$*. Then*

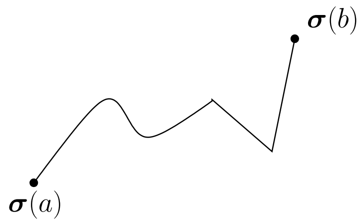
$$\int_{\boldsymbol{\sigma}} f dr = \int_{\boldsymbol{\rho}} f dr.$$

Let $\mathbf{F}$ *be a vector field continuous on the* C^1 *path* $\boldsymbol{\sigma}\colon [a_1,b_1] \to \mathbb{R}^3$ *and let* $\boldsymbol{\rho}\colon [a,b] \to \mathbb{R}^3$ *be a reparametrization of* $\boldsymbol{\sigma}$*, then*

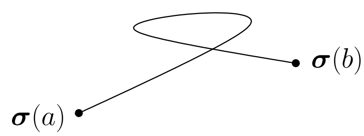
$$\begin{aligned}\int_{\boldsymbol{\rho}} \mathbf{F}\cdot d\mathbf{r} &= +\int_{\boldsymbol{\sigma}} \mathbf{F}\cdot d\mathbf{r}, \text{ if } \boldsymbol{\rho} \text{ is orientation preserving,}\\ &= -\int_{\boldsymbol{\sigma}} \mathbf{F}\cdot d\mathbf{r}, \text{ if } \boldsymbol{\rho} \text{ is orientation reversing.}\end{aligned}$$

The previous theorem also holds for piecewise C^1 paths by breaking up the intervals into segments on which the paths are of class C^1 and summing the integrals over separate intervals.

Definition 4.6. *We define a* **simple curve** C *to be the image of a piecewise* C^1 *map* $\boldsymbol{\sigma}\colon I = [a,b] \to \mathbb{R}^3$ *that is one-to-one on an interval* I*. Thus, it is a curve that does not intersect itself.*



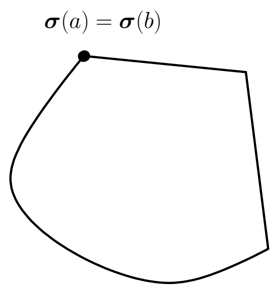
Simply curve



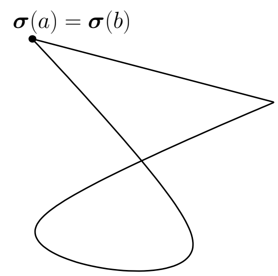
Nonsimply curve

$\boldsymbol{\sigma}(a)$ *and* $\boldsymbol{\sigma}(b)$ *are called the endpoints of the curve. Each simple curve* C *has two orientation or directions associated with it. The simple curve* C *together with a sense of direction is called an* **oriented simple curve.**

Definition 4.7. *A* **simple closed curve** C *is the image of a piecewise* C^1 *map* $\boldsymbol{\sigma}\colon I = [a,b] \to \mathbb{R}^3$ *that is one-to-one on* $[a,b)$ *and satisfies* $\boldsymbol{\sigma}(a) = \boldsymbol{\sigma}(b)$*.*



Simply curve



Nonsimply curve

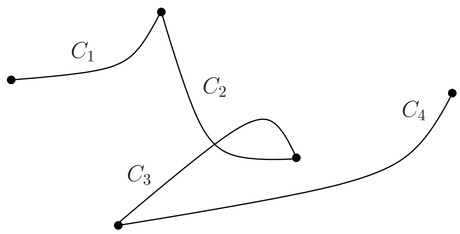
Simple closed curves have two possible orientations: counterclockwise or positive and clockwise or negative.

Notation: If C is a closed curve, then $\int_{\boldsymbol{\sigma}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\boldsymbol{\sigma}} \mathbf{F} \cdot d\mathbf{r} \to$ is called also the circulation of $\mathbf{F}$ around C.

If C is an oriented simple curve or an oriented closed curve, we define

$$\begin{aligned} \int_C f\,dr &= \int_{\boldsymbol{\sigma}} f\,dr, \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\boldsymbol{\sigma}} \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

Where $\boldsymbol{\sigma}$ is any orientation-preserving parametrization of C.



If C is a curve (or an oriented curve) that is made up of several (oriented) component curves C_i, $i = 1, \cdots, k$, we can parametrize C by parametrizing the pieces $C_1, , \cdots, C_k$ separately. Then, our integral will be

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{C_k} \mathbf{F} \cdot d\mathbf{r}.$$

One reason for writing a curve as a sum of components is that it may be easier to parametrize the components C_i individually than it is to do it as a whole.

Conservative Fields

Recall that a vector field $\mathbf{F}$ is a **gradient vector field** if $\mathbf{F} = \boldsymbol{\nabla} f$ for some real-valued function f. Thus, in $\mathbb{R}^3$ this reads as $\mathbf{F} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$. We call $\mathbf{F}$ a **conservative vector field** and f is called the **potential** of $\mathbf{F}$.

We have the following generalization of the Fundamental Theorem of Calculus:

Theorem 4.8. *Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and that $\boldsymbol{\sigma}\colon [a,b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then*

$$\int_{\boldsymbol{\sigma}} \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(\boldsymbol{\sigma}(b)) - f(\boldsymbol{\sigma}(a)).$$

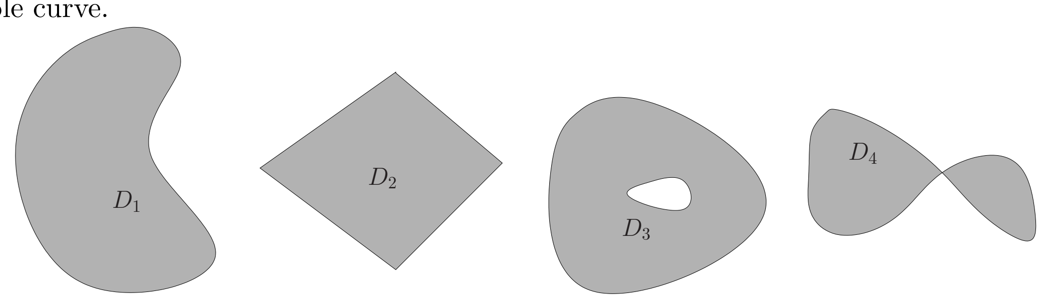
Note. if $\mathbf{F} = -\boldsymbol{\nabla} U$ is a conservative force field, the work done by $\mathbf{F}$ on a particle does depend only on the endpoints of the curve and it does not depend on the trajectory:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \boldsymbol{\nabla} U \cdot d\mathbf{r} = U(\boldsymbol{\sigma}(a)) - U(\boldsymbol{\sigma}(b)).$$

Where U denotes the potential energy: gravitational, electric or elastic potential energy.

Definition 4.9. *Let $D \subset \mathbb{R}^n$, D is* **simply connected** *if every closed curve on D can be contracted continuously to a point.*

A simply connected set on $\mathbb{R}^2$ is one whose boundary is formed by a single closed simple curve.



Simply connected sets Nonsimply connected sets

Theorem 4.10. *Let $D \subset \mathbb{R}^n$, b simply connected and $\mathbf{F}$ a C^1 vector field defined on D. The following are equivalent:*

1. $\mathbf{F}$ *is a* **conservative vector field** *on D, that is, $\exists f \in C^2(D)$ such that $\mathbf{F} = \nabla f$. Where f is called the potential of $\mathbf{F}$.*

2. *For every closed curve $\boldsymbol{\sigma}$ on D,*

$$\int_{\boldsymbol{\sigma}} \mathbf{F} \cdot d\mathbf{r} = 0.$$

3. *For every $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ curves on D with same endpoints, we have*

$$\int_{\boldsymbol{\sigma_1}} \mathbf{F} \cdot d\mathbf{r} = \int_{\boldsymbol{\sigma_2}} \mathbf{F} \cdot d\mathbf{r}.$$

4. $n = 2$. *If $\mathbf{F} = (P, Q)$ then* $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

4'. $n = 3$. $\nabla \times \mathbf{F} = 0$.

Remark: The **curl** of a vector field $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$, If $\mathbf{F} = (F_1, F_2, F_3) \in C^1$ is

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix},$$

with the property: $\nabla \times (\nabla f) = 0$.