Universidad Carlos III de Madrid
Calculus II

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Integrals depending on a parameter

Def.

Let $f\colon [a,b]\times[c,d]\to\mathbb{R}$, if for each fixed $t\in[c,d]$ the function $f(x,t)$ is integrable over $[a,b]$ on the x variable, we define the following function $F\colon [a,b]\to\mathbb{R}$ as

$$F(t)=\int_a^b f(x,t)dx.$$

We call $F(t)$ **an integral depending on a parameter.**

Theorem

f continuous on $[a,b]\times[c,d] \Rightarrow F$ continuous on $[c,d]$.

Theorem

f and $f_t=\dfrac{\partial f}{\partial t}$ continuous on $[a,b]\times[c,d] \Rightarrow F$ differentiable on $[c,d]$

$$F'(t)=\int_a^b f_t(x,t)dx$$

.

Theorem (Leibniz's Theorem)

Let f and $f_t = \dfrac{\partial f}{\partial t}$ be continuous functions on $[a,b] \times [c,d]$ and α, β differentiable functions on $[c,d]$ with image on $[a,b]$, that is, $\alpha(t), \beta(t) \colon [c,d] \to [a,b]$, $x \in [\alpha(t), \beta(t)] \subset [a,b]$. We define

$$G(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx,$$

then G is differentiable on $[c,d]$ and

$$G'(t) = f(\beta(t),t) \cdot \beta'(t) - f(\alpha(t),t) \cdot \alpha'(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x,t)dx$$

The previows theorems cannot be used when the integrals are improper. So we have the following results that are valid if the integrals are improper or not.

Theorem

Let $f(x,t)$ *be a cont. function of* t *on* $[c,d]$ *for almost every* $x \in [a,b]$*, if there exists* $g(x)$ *integrable on* $[a,b]$ *s. t.* $|f(x,t)| \leq g(x), \forall t \in [c,d]$ *and for almost every* $x \in [a,b]$*, then* F **is continuous** *on* $[c,d]$.

Theorem

Let $f_t(x,t)$ *be a cont. function of* t *on* $[c,d]$ *for almost every* $x \in [a,b]$*, if there exists* $g(x)$ *integrable on* $[a,b]$ *s. t.* $|f_t(x,t)| \leq g(x), \forall t \in [c,d]$ *and for almost every* $x \in [a,b]$*, then* F **is differentiable** *on* $[c,d]$ *and*

$$F'(t) = \int_a^b f_t(x,t)dx$$

.

Theorem

$f(x,t)$ *be integrable on* $[a,b] \times [c,d] \Rightarrow F(t)$ *is integrable on* $[c,d]$*, and*

$$\int_c^d F(t)dt = \int_c^d \int_a^b f(x,t)dxdt = \int_a^b \int_c^d f(x,t)dtdx.$$

The Gamma Function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0$$

It is a generalization of the factorial function to real and complex numbers.

Properties of the Gamma function

1. $\Gamma(x)$ is continuous and differentiable.
2. $\Gamma(x) \in C^{\infty}$ and $\Gamma^{n)}(x) = \int_0^{\infty} (\log t)^n t^{x-1} e^{-t} dt.$
3. $\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(1/2) = \sqrt{\pi}.$
4. $\Gamma(x+1) = x\Gamma(x).$
5. $\Gamma(n+1) = n!, \ n \in \mathbb{N}.$
6. $\lim_{x \to 0^+} \Gamma(x) = \infty.$

The Beta function

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad p,q>0$$

1. $B(p,q)=B(q,p)$.
2. $B(p,q)$ is continuous and differentiable on each variable.
3. $B(p,q)\in C^{\infty}$.
$$\frac{\partial^{n+m}}{\partial p^n \partial q^m}B(p,q)=\int_0^1 x^{p-1}(\log x)^n(1-x)^{q-1}(\log(1-x))^m dx, \ \ p,q>0.$$
4. $B(p,q)=\dfrac{q-1}{p+q-1}B(p,q-1), \quad q>1.$
5. $B(m+1,n+1)=\dfrac{1}{m+n+1}\begin{pmatrix} m+n \\ n \end{pmatrix}^{-1}, \ \ m,n\in\mathbb{N}.$
6. $B(p,q)=2\displaystyle\int_0^{\pi/2}(\cos t)^{2p-1}(\sin t)^{2q-1}dt.$
7. $B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$
8. $B(p,q)=\displaystyle\int_0^{\infty}\frac{t^{p-1}}{(1+t)^{p+q}}dt.$
9. $B(1/2,1/2)=\pi.$

Def.

Let $f\colon [0,\infty) \to \mathbb{R}$ be integrable, with exponential growth, that is, $|f(x)| \leq Ce^{\alpha x}, \forall x > T$, where C, α, T are constants depending of f, we define the **Laplace Transform of** f as

$$L[f(x)](s) \equiv F(s) = \int_0^\infty e^{-sx} f(x)dx.$$

Properties

1. $L[f(x)](s)$ converges for $s \in (\alpha, \infty)$ and is continuous on (α, ∞).
2. $L[f(x)](s) \leq \dfrac{C}{s-\alpha}, \quad s > \alpha.$
3. $L[\alpha f(x) + \beta g(x)](s) = \alpha L[f(x)](s) + \beta L[g(x)](s).$
4.

$$L[1](s) = \frac{1}{s},\ s > 0. \qquad L[e^{ax}](s) = \frac{1}{s-a},\ s > a.$$

$$L[x^n](s) = \frac{n!}{s^{n+1}},\ s > 0,\ n \in \mathbb{N}. \qquad L[x^\alpha](s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, s > 0, \alpha > -1.$$

$$L[\sin(ax)](s) = \frac{a}{s^2+a^2},\ s > 0. \qquad L[\cos(ax)](s) = \frac{s}{s^2+a^2},\ s > 0.$$

Properties

5. Translation: $L[e^{-ax}f(x)](s) = L[f(x)](s+a) = F(s+a),\ a \in \mathbb{R}.$
6. Redef. $f(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases} \Rightarrow$

   $L[f(x-a)](s) = e^{-as}L[f(x)](s),\ a > 0.$
7. $L[f(ax)](s) = \frac{1}{a}L[f(x)]\left(\frac{s}{a}\right),\ a > 0.$
8. $\frac{d^n}{ds^n}L[f(x)](s) = (-1)^n L[x^n f(x)](s),\ n = 1, 2, 3\ldots$
9. Let $f^{(n-1)}$ be differentiable on $(0, \infty)$ and $f, f', \ldots, f^{(n-1)}$ continuous on $x \geq 0$, then

   $L[f^{(n)}(x)](s) = s^n L[f(x)](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots . - f^{(n-1)}(0).$

   In particular

$$\begin{aligned} L[f'(x)](s) &= sL[f(x)](s) - f(0), \\ L[f''(x)](s) &= s^2 L[f(x)](s) - sf(0) - f'(0). \end{aligned}$$

Def. (The convolution of $f(x)$ and $g(x)$)

$$f \star g(x) = \int_0^x f(u)g(x-u)du, \text{ def. for } x \geq 0.$$

Properties: $\begin{cases} f \star g(x) = g \star f(x). \\ L[f \star g(x)](s) = F(s)G(s). \end{cases}$

Def. (The Inverse Laplace Transform of $F(s)$)

Given a function $F(s)$, if there exists $f(x)$, continuous on $x \geq 0$, such that $L[f(x)](s) = F(s)$, we define it as $L^{-1}[F(s)](x) = f(x)$.

Application of The Transform of Laplace

Differential equation for $f(x)$

$\Downarrow L$

Transformed equation for $F(s)$

$\Downarrow Solve$

$F(s)$

$\Downarrow L^{-1}$

The solution is $f(x)$