Universidad Carlos III de Madrid
Calculus II

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Surface Integrals

DEF.

A parametrized surface is a function $\phi\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$, $\phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$.
The surface S corresponding to the function ϕ is its image: $S = \phi(D)$.
ϕ differentiable or C^1 $\to$ S differentiable or C^1 surface.

ϕ differentiable at $(u_0, v_0) \in \mathbb{R}^2$ $\to$

$$\mathbf{D}_v\phi(u_0,v_0) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\Bigg|_{(u_0,v_0)}.$$

$$\mathbf{D}_u\phi(u_0,v_0) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\Bigg|_{(u_0,v_0)}.$$

Denote $\mathbf{T}_v := \mathbf{D}_v\phi(u,v)$ and $\mathbf{T}_u := \mathbf{D}_u\phi(u,v)$.

Def.

A surface S is said to be **smooth** at $\phi(u_0, v_0)$ if $\mathbf{T}_u \times \mathbf{T}_v \neq 0$ at (u_0, v_0). The surface is called **smooth** if it is smooth at all points $\phi(u_0, v_0) \in S$. The nonzero vector $\mathbf{T}_u \times \mathbf{T}_v$ is normal to S at each point.

Def.

If a parametrized surface $\phi\colon D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ is smooth at $\phi(u_0, v_0)$ we define the **tangent plane** of the surface at $\phi(u_0, v_0)$ as the plane determined by the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$. Therefore, $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ is a normal vector, and the equation of the plane will be

$$(x - x_0, y - y_0, z - z_0) \cdot \mathbf{n} = 0,$$

where $\mathbf{n}$ is evaluated at (u_0, v_0) and $(x_0, y_0, z_0) = \phi(u_0, v_0)$.

We will consider piecewise smooth surfaces that are unions of images of parametrized surfaces $\phi_\mathbf{i}\colon D_i \subset \mathbb{R}^2 \to S_i \subset \mathbb{R}^3$ for which:

- D_i is an elementary region in the plane.
- $\phi_\mathbf{i}$ is C^1 and one-to-one.
- The image of $\phi_\mathbf{i}$, S_i, is smooth, except at a finite number of points.

Def. (Integral of a scalar function over a surface)

Let $f(x,y,z)\colon \mathbb{R}^3 \to \mathbb{R}$ be a real-valued continuous function defined on a surface S, parametrized by $\phi\colon D \to S$, $\phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$. We define the integral of f over S as

$$\iint_S f(x,y,z)ds = \iint_D f(\phi(u,v))\|\mathbf{T}_u \times \mathbf{T}_v\| dudv.$$

Note. $Area(S) = \iint_S 1ds.$

Note. If S is given as the union of several surfaces that do not intersect, $S = \cup_{i=1}^n S_i$, then $\iint_S fds = \sum_{i=1}^n \iint_{S_i} fds.$

Integral of Vector Functions Over Surfaces

Def. (Integral of a vector field over a surface)

Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field defined over S, the image of a parametrized surface
$\phi\colon D \to S$. The **surface integral** of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_D \mathbf{F}(\phi(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \, dudv.$$

Def. (An oriented surface)

is a two-sided surface with one side specified as the **outside or positive side**. The other side is called the **inside or negative side**.

A side of a surface S $\longrightarrow$ choose a unit normal vector $\mathbf{n}$ pointing away from the positive side of S at each point. A parametrization $\phi\colon D \to S$ is

- **orientation-preserving** if $\dfrac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \mathbf{n}\big(\phi(u,v)\big)$,
- **orientation-reversing** if $\dfrac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = -\mathbf{n}\big(\phi(u,v)\big)$,

for all $(u,v) \in D$ for which S is smooth at $\phi(u,v)$.

Def.

For an oriented smooth surface S and any orientation-preserving parametrization ϕ of the surface, we define the **surface integral of** $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$, a continuous vector field defined over S or the **flux of F across the surface** S as

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_\phi \mathbf{F} \cdot d\mathbf{s}.$$

The flux of $\mathbf{F}$ across S measures the amount of the vector field $\mathbf{F}$ that flows across the surface per unit time. It can be written also as

$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_S \mathbf{F} \cdot \mathbf{n}\, ds$, where $\mathbf{F} \cdot \mathbf{n}$ is the normal component of $\mathbf{F}$ over S.

Note. If we have another parametrization ψ that is orientation-reversing then

$$\iint_\psi \mathbf{F} \cdot d\mathbf{s} = -\iint_\phi \mathbf{F} \cdot d\mathbf{s}.$$