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Unit 5

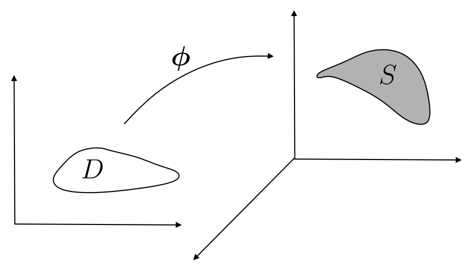
Surface Integrals

Integral of Scalar Functions Over Surfaces

Definition 5.1. A parametrized surface *is a function* $\boldsymbol{\phi}\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$,
$\boldsymbol{\phi}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$.

The surface S *corresponding to the function* $\boldsymbol{\phi}$ *is its image:* $S = \boldsymbol{\phi}(D)$.

If $\boldsymbol{\phi}$ *is differentiable or it is of class* C^1, *we call* S *a differentiable or a* C^1 *surface, respectively.*



If $\boldsymbol{\phi}$ is differentiable at $(u_0, v_0) \in \mathbb{R}^2$, fixing u as u_0, we get a map $\boldsymbol{\phi}(u_0, t)$, whose image is a curve on the surface. The tangent vector to this curve at $\boldsymbol{\phi}(u_0, v_0)$ is given by

$$\mathbf{D}_v\boldsymbol{\phi}(u_0, v_0) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\bigg|_{(u_0,v_0)}.$$

Similarly, fixing $v = v_0$ we obtain the tangent vector to the curve $\boldsymbol{\phi}(t, v_0)$,

$$\mathbf{D}_u\boldsymbol{\phi}(u_0, v_0) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\bigg|_{(u_0,v_0)}.$$

Let us denote the tangent vectors as $\mathbf{T}_v := \mathbf{D}_v\boldsymbol{\phi}(u,v)$ and $\mathbf{T}_u := \mathbf{D}_u\boldsymbol{\phi}(u,v)$.

Definition 5.2. *A surface* S *is said to be* **smooth** *at* $\boldsymbol{\phi}(u_0, v_0)$ *if* $\mathbf{T}_u \times \mathbf{T}_v \neq 0$ *at* (u_0, v_0). *The surface is called* **smooth** *if it is smooth at all points* $\boldsymbol{\phi}(u_0, v_0) \in S$. *The nonzero vector* $\mathbf{T}_u \times \mathbf{T}_v$ *is normal to* S *at each point.*

Definition 5.3. *If a parametrized surface* $\boldsymbol{\phi}\colon D\subset\mathbb{R}^2\to S\subset\mathbb{R}^3$ *is smooth at* $\boldsymbol{\phi}(u_0,v_0)$ *we define the* **tangent plane** *of the surface at* $\boldsymbol{\phi}(u_0,v_0)$ *as the plane determined by the vectors* $\mathbf{T}_u$ *and* $\mathbf{T}_v$. *Therefore,* $\mathbf{n}=\mathbf{T}_u\times\mathbf{T}_v$ *is a normal vector, and the equation of the plane will be*

$$(x-x_0,y-y_0,z-z_0)\cdot\mathbf{n}=0,$$

where $\mathbf{n}$ *is evaluated at* (u_0,v_0) *and* $(x_0,y_0,z_0)=\boldsymbol{\phi}(u_0,v_0)$.

Property: If the surface S is given as the graph of a differentiable function $g\colon R^2\to\mathbb{R}$, then S is differentiable at every point, so it is smooth and, therefore, has a tangent plane everywhere.

We will consider piecewise smooth surfaces that are unions of images of parametrized surfaces $\boldsymbol{\phi_i}\colon D_i\subset\mathbb{R}^2\to S_i\subset\mathbb{R}^3$ for which:

- D_i is an elementary region in the plane.
- $\boldsymbol{\phi_i}$ is C^1 and one-to-one.
- The image of $\boldsymbol{\phi_i}$, S_i, is smooth, except possibly at a finite number of points.

Definition 5.4 (Integral of a scalar function over a surface)**.**
Let $f(x,y,z)\colon\mathbb{R}^3\to\mathbb{R}$ *be a real-valued continuous function defined on a surface* S, *parametrized by* $\boldsymbol{\phi}\colon D\to S$, $\boldsymbol{\phi}(u,v)=\big(x(u,v),y(u,v),z(u,v)\big)$. *We define the integral of* f *over* S *as*

$$\iint\limits_S f(x,y,z)ds=\iint\limits_D f(\boldsymbol{\phi}(u,v))\|\mathbf{T}_u\times\mathbf{T}_v\|dudv.$$

Note. $Area(S)=\iint\limits_S 1ds.$

Note. If S is given as the union of several surfaces that do not intersect, $S=\cup_{i=1}^n S_i$, then $\iint\limits_S fds=\sum_{i=1}^n\iint\limits_{S_i} fds.$

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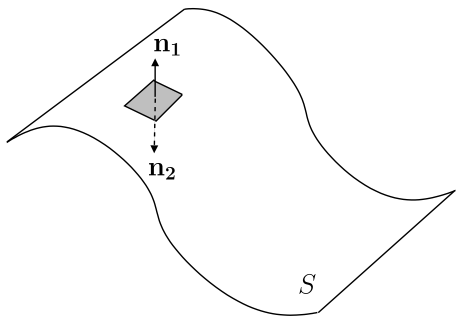
Integral of Vector Functions Over Surfaces

Definition 5.5 (Integral of a vector field over a surface)**.** *Let* $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ *be a continuous vector field defined over* S*, the image of a parametrized surface* $\boldsymbol{\phi}\colon D \to S$*. The* **surface integral** *of* $\mathbf{F}$ *over* S *is*

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_D \mathbf{F}(\boldsymbol{\phi}(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v du dv.$$

Definition 5.6. *An* **oriented surface** *is a two-sided surface with one side specified as the* **outside or positive side***. The other side is called the* **inside or negative side***.*



At each point in S there are two unit normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$, where $\mathbf{n_2} = -\mathbf{n_1}$. We can associate each of these normals with one side of the surface. Thus, to specify a side of a surface S we choose a unit normal vector $\mathbf{n}$ pointing away from the positive side of S at each point. We say that a parametrization $\boldsymbol{\phi}\colon D \to S$ of the surface is

- **orientation-preserving** if $\dfrac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = \mathbf{n}\big(\boldsymbol{\phi}(u,v)\big)$,
- **orientation-reversing** if $\dfrac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} = -\mathbf{n}\big(\boldsymbol{\phi}(u,v)\big)$,

for all $(u,v) \in D$ for which S is smooth at $\boldsymbol{\phi}(u,v)$.

Definition 5.7. *For an oriented smooth surface* S *and any orientation-preserving parametrization* $\boldsymbol{\phi}$ *of the surface, we define the* **surface integral of F**: $\mathbb{R}^3 \to \mathbb{R}^3$*, a continuous vector field defined over* S *or the* **flux of F across the surface** S *as*

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_{\boldsymbol{\phi}} \mathbf{F} \cdot d\mathbf{s}.$$

The flux of $\mathbf{F}$ across S measures the amount of the vector field $\mathbf{F}$ that flows across the surface per unit time. It can be written also as

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_S \mathbf{F} \cdot \mathbf{n}\, ds,$$ where $\mathbf{F} \cdot \mathbf{n}$ is the normal component of $\mathbf{F}$ over S.

Note. If we have another parametrization $\boldsymbol{\psi}$ that is orientation-reversing then

$$\iint_{\boldsymbol{\psi}} \mathbf{F} \cdot d\mathbf{s} = -\iint_{\boldsymbol{\phi}} \mathbf{F} \cdot d\mathbf{s}.$$