# 1. INTEGRATION IN ONE VARIABLE

**Problem 1.1** Find the following antiderivatives:

1. $\int x\,\mathrm{tg}^2(2x)\,dx$
2. $\int \mathrm{tg}^3 x \sec^4 x\,dx$
3. $\int \frac{\sqrt{x}+1}{x+3}\,dx$
4. $\int \frac{(x+1)^3}{\sqrt{1-(x+1)^2}}\,dx$
5. $\int \frac{x^2}{(x-1)^3}\,dx$
6. $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$
7. $\int \frac{\sin^2 x \cos^5 x}{\mathrm{tg}^3 x}\,dx$
8. $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$
9. $\int \mathrm{e}^x \sin \pi x\,dx$
10. $\int \frac{dx}{\cos^4 x}$
11. $\int \sin^2 x\,dx$
12. $\int \sin^4 x\,dx$
13. $\int \cos^2 x\,dx$
14. $\int \cos^6 x\,dx$
15. $\int \sin^2 x \cos^2 x\,dx$
16. $\int \frac{dx}{3+\sqrt{2x+5}}$
17. $\int \sqrt{\frac{x-1}{x+1}}\,dx$
18. $\int \mathrm{arc\,tg}\,\sqrt[3]{x}\,dx$
19. $\int \sqrt{\sqrt{x}+1}\,dx$
20. $\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$
21. $\int \sqrt{2+\mathrm{e}^x}\,dx$
22. $\int \mathrm{e}^{\sin x} \cos^3 x\,dx$
23. $\int \sin^5 x\,dx$
24. $\int \cos^3 x \sin^2 x\,dx$
25. $\int \mathrm{tg}^2 x\,dx$
26. $\int \mathrm{tg}^3 x\,dx$
27. $\int x^3 \sqrt{1-x^2}\,dx$
28. $\int \frac{\sin x + 3\cos x}{\sin x \cos x + 2\sin x}\,dx$
29. $\int \frac{\sin x + 3\cos x}{\sin x + 2\cos x}\,dx$
30. $\int \mathrm{tg}^2(3x)\sec^3(3x)\,dx$
31. $\int \frac{4x^4 - x^3 - 46x^2 - 20x + 153}{x^3 - 2x^2 - 9x + 18}\,dx$
32. $\int \cos(\log x)\,dx$
33. $\int \frac{\mathrm{e}^{4x}}{\mathrm{e}^{2x}+\mathrm{e}^x+2}\,dx$
34. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$
35. $\int \frac{x^2}{(x^2+1)^{5/2}}\,dx$
36. $\int \frac{2}{x^2-2x+2}\,dx$
37. $\int \frac{dx}{\cos^2 x}$
38. $\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$
39. $\int \frac{x}{(x^2+1)^{5/2}}\,dx$
40. $\int x^2(1-x^2)^{-3/2}\,dx$
41. $\int \sqrt{\mathrm{e}^x-1}\,dx$
42. $\int \frac{2x^2+3}{x^2(x-1)}\,dx$

43. $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$
44. $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$
45. $\displaystyle\int x^2\sqrt{x-1}\,dx$
46. $\displaystyle\int \sec^6 x\,dx$
47. $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$
48. $\displaystyle\int \frac{dx}{e^x-4e^{-x}}$
49. $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$
50. $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$
51. $\displaystyle\int e^x\cos 2x\,dx$
52. $\displaystyle\int x^2\log x\,dx$
53. $\displaystyle\int \sin^3 x\cos^2 x\,dx$
54. $\displaystyle\int \cos^4 x\,dx$
55. $\displaystyle\int \mathrm{tg}^4 x\,dx$
56. $\displaystyle\int \sec^3 x\,dx$
57. $\displaystyle\int \frac{dx}{1-\sin x}$
58. $\displaystyle\int \sin(\log x)\,dx$
59. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$
60. $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$
61. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$
62. $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$
63. $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$
64. $\displaystyle\int \frac{dx}{\sqrt[3]{(1-2x)^2}-\sqrt{1-2x}}$
65. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$
66. $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$
67. $\displaystyle\int x^m\log x\,dx$
68. $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$
69. $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$
70. $\displaystyle\int \cos^2(\log x)\,dx$
71. $\displaystyle\int (\log x)^3\,dx$
72. $\displaystyle\int x(\log x)^2\,dx.$

Hint: IBP means integration by parts and CV change of variables.

1. IBP $dv=\mathrm{tg}^2(2x)dx=(\sec^2(2x)-1)dx$.
2. CV $t=\mathrm{tg}\ x$.
3. CV $t=\sqrt{x}$.
4. CV $t=\sqrt{1-(x+1)^2}$.
5. Do partial fraction decomposition or expand $x^2$ in powers of $x-1$.
6. CV $x=\sec t$.
7. As the integrand is odd in sine, CV $t=\cos x$.
8. The derivative of the denominator almost appears in the numerator.
9. IBP twice using $dv=e^x dx$.
10. As the integrand is even in sine and cosine, CV $t=\mathrm{tg}\,x$.
11, 12, 13, 14 and 15. Use double angle formulas.
16. CV $t=3+\sqrt{2x+5}$ or $t=\sqrt{2x+5}$.
17. CV $t=\sqrt{(x-1)/(x+1)}$.
18. CV $x=t^3$, after do IBP with $u=\mathrm{arc\,tg}\,t$.
19. CV $t=\sqrt{\sqrt{x}+1}$.
20. CV $t=\sqrt{x+2}$.
21. CV $t=\sqrt{e^x+2}$.
22. CV $t=\sin x$, after do IBP twice with $dv=e^t dt$.
23. As the integrand is odd in sine, CV $t=\cos x$.
24. As the integrand is odd in cosine, CV $t=\sin x$.
25. $\mathrm{tg}^2 x=\sec^2 x-1$, or apply the CV $t=\mathrm{tg}\,x$.
26. CV $t=\mathrm{tg}\,x$.
27. CV $t=\sqrt{1-x^2}$.
28. CV $t=\mathrm{tg}(x/2)$.
29. CV $t=\mathrm{tg}\,x$.
30. CV $t=\sin(3x)$.
31. The denominator is $(x-2)(x-3)(x+3)$.

32. IBP twice using $dv = dx$ or use the CV $t = \log x$.
33. CV $t = \mathrm{e}^x$.
34. CV $t = \sqrt{1 + x^{1/3}}$.
35. CV $x = \operatorname{tg} t$.
36. The denominator is equal to $(x-1)^2 + 1$.
37. It is immediate.
38. CV $x + 2 = t^3$.
39. It is immediate. Can be integrated also using the CV $t = x^2 + 1$.
40. CV $x = \sin t$.
41. CV $t = \sqrt{\mathrm{e}^x - 1}$.
42. Decompose in partial fractions.
43. It is immediate. Can be integrated also using the CV $t = \sqrt{1 - \sqrt{x}}$.
44. Multiply and divide by $1 - \cos x$.
45. IBP twice taking the derivative of the polynomial or use the CV $t = \sqrt{x - 1}$.
46. CV $t = \operatorname{tg} x$.
47. CV $t = 1 + x^2$.
48. CV $t = \mathrm{e}^x$.
49. CV $t^2 = 1 + x$.
50. CV $t^3 = 1 - x$.
51. IBP twice using $dv = \mathrm{e}^x dx$.
52. IBP $u = \log x$.
53. CV $t = \cos x$.
54. Use double angle formulas.
55. CV $t = \operatorname{tg} x$.
56. CV $t = \sin x$.
57. Multiply and divide by $1 + \sin x$.
58. IBP twice using $dv = dx$ or use the CV $t = \log x$.
59. CV $x = \sin t$.
60. It is immediate (you can also use the CV $t^2 = 1 + x^2$).
61. CV $t^2 = \mathrm{e}^{2x} - 1$.
62. CV $t = \mathrm{e}^x$.
63. The denominator is equal to $(x-1)^2(x+1)^2$.
64. CV $t^6 = 1 - 2x$.
65. CV $x = 3 \sin t$.
66. $x^2 + x + 1 = (x + 1/2)^2 + 3/4$.
67. IBP $u = \log x$.
68. CV $t = \sin x$.
69. CV $t^2 = x^3$.
70. Use double angle formulas. Next do IBP twice using $dv = dx$ or use the CV $t = 2 \log x$.
71. IBP $u = (\log x)^3$.
72. IBP $u = (\log x)^2$.

*Solution:*
1. $\frac{1}{2}\, x \operatorname{tg}(2x) + \frac{1}{4}\, \log|\cos(2x)| - \frac{1}{2}\, x^2 + c$.
2. $\frac{1}{6}\, \operatorname{tg}^6 x + \frac{1}{4}\, \operatorname{tg}^4 x + c$.
3. $2\sqrt{x} + \log|x+3| - 2\sqrt{3}\, \operatorname{arc\,tg} \sqrt{\frac{x}{3}} + c$.
4. $\frac{1}{3}\,(1 - (x+1)^2)^{3/2} - (1 - (x+1)^2)^{1/2} + c$.
5. $\frac{-1}{2(x-1)^2} - \frac{2}{x-1} + \log|x-1| + c$.
6. $\frac{1}{2}\, x\sqrt{x^2-1} + \frac{3}{2}\, \log|x + \sqrt{x^2-1}\,| + c$.
7. $\frac{1}{7}\, \cos^7 x + \frac{1}{5}\, \cos^5 x + \frac{1}{3}\, \cos^3 x + \cos x + \frac{1}{2}\, \log(1-\cos x) - \frac{1}{2}\, \log(1+\cos x) + c$.
8. $-\log|\sin x + \cos x| + c$.
9. $\frac{1}{1+\pi^2}\, \mathrm{e}^x(\sin \pi x - \pi \cos \pi x) + c$.
10. $\frac{1}{3}\, \operatorname{tg}^3 x + \operatorname{tg} x + c$.
11. $\frac{1}{2}\, x - \frac{1}{4}\, \sin 2x + c$.
12. $\frac{3}{8}\, x - \frac{1}{4}\, \sin 2x + \frac{1}{32}\, \sin 4x + c$.
13. $\frac{1}{2}\, x + \frac{1}{4}\, \sin 2x + c$.
14. $\frac{5}{16}\, x + \frac{1}{4}\, \sin 2x + \frac{3}{64}\, \sin 4x - \frac{1}{48}\, \sin^3 2x + c$.
15. $\frac{1}{8}\, x - \frac{1}{32}\, \sin 4x + c$.
16. $\sqrt{2x+5} - 3\, \log\left(3 + \sqrt{2x+5}\,\right) + c$.
17. $\frac{-1}{t-1} + \log|t-1| - \frac{1}{t+1} - \log|t+1| + c$, where $t = \sqrt{(x-1)/(x+1)}$.
18. $x \operatorname{arc\,tg}(x^{1/3}) - \frac{1}{2}\, x^{2/3} + \frac{1}{2}\, \log(x^{2/3} + 1) + c$.
19. $\frac{4}{5}\, (\sqrt{x} + 1)^{5/2} - \frac{4}{3}\, (\sqrt{x} + 1)^{3/2} + c$.
20. $x - 2\sqrt{x+2} + 2\log(\sqrt{x+2} + 1) + c$.
21. $2\sqrt{2 + \mathrm{e}^x} + \sqrt{2}\, \log(\sqrt{2+\mathrm{e}^x} - \sqrt{2}\,) - \sqrt{2}\, \log(\sqrt{2+\mathrm{e}^x} + \sqrt{2}\,) + c$.

22. $-(1-\sin x)^2\, e^{\sin x} + c.$
23. $\frac{-1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + c.$
24. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c.$
25. $\operatorname{tg} x - x + c.$
26. $\frac{1}{2}\operatorname{tg}^2 x + \log|\cos x| + c.$
27. $\frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + c.$
28. $\log\left|\operatorname{tg}\frac{x}{2}\right| - 2\log\left(\operatorname{tg}^2\frac{x}{2}+3\right) + \frac{2}{\sqrt{3}}\operatorname{arc\,tg}\frac{\operatorname{tg}(x/2)}{\sqrt{3}} + c.$
29. $\frac{1}{5}\log|\operatorname{tg} x + 2| - \frac{1}{10}\log\left(\operatorname{tg}^2 x + 1\right) + \frac{7}{5}x + c.$
30. $\frac{1}{12}\sec^3 3x \operatorname{tg} 3x - \frac{1}{24}\sec 3x \operatorname{tg} 3x - \frac{1}{24}\log|\sec 3x + \operatorname{tg} 3x| + c.$
31. $2x^2 + 7x + 3\log|x-2| - 4\log|x-3| + 5\log|x+3| + c.$
32. $\frac{1}{2}x\cos(\log x) + \frac{1}{2}x\sin(\log x) + c.$
33. $\frac{1}{2}e^{2x} - e^x - \frac{1}{2}\log\left(e^{2x}+e^x+2\right) + \frac{5}{\sqrt{7}}\operatorname{arc\,tg}\frac{2e^x+1}{\sqrt{7}} + c.$
34. $\frac{6}{5}(1+x^{1/3})^{5/2} - 2(1+x^{1/3})^{3/2} + c.$
35. $\frac{1}{3}x^3(x^2+1)^{-3/2} + c.$
36. $2\operatorname{arc\,tg}(x-1) + c.$
37. $\operatorname{tg} x + c.$
38. $\log\left|(x+2)^{1/3}-1\right| - \frac{1}{2}\log\left((x+2)^{2/3}+(x+2)^{1/3}+1\right) + \sqrt{3}\operatorname{arc\,tg}\frac{2(x+2)^{1/3}+1}{\sqrt{3}} + c.$
39. $\frac{-1}{3}(x^2+1)^{-3/2} + c.$
40. $x(1-x^2)^{-1/2} - \arcsin x + c.$
41. $2\sqrt{e^x-1} - 2\operatorname{arc\,tg}\sqrt{e^x-1} + c.$
42. $5\log|x-1| - 3\log|x| + \frac{3}{x} + c.$
43. $\frac{-4}{3}(1-\sqrt{x})^{3/2} + 2\sqrt{x} + c.$
44. $\csc x - \cot x - \log|\csc x + \cot x| - \log|\sin x| = \csc x - \cot x - \log(1+\cos x) + c.$
45. $\frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c.$
46. $\frac{1}{5}\operatorname{tg}^5 x + \frac{2}{3}\operatorname{tg}^3 x + \operatorname{tg} x + c.$
47. $\frac{-1}{2(1+x^2)} + \frac{1}{4(1+x^2)^2} + c.$
48. $\frac{1}{4}\log|e^x-2| - \frac{1}{4}\log(e^x+2) + c.$
49. $2\operatorname{arc\,tg}\sqrt{1+x} + c.$
50. $\frac{-3}{2}(1-x)^{2/3} + 3(1-x)^{1/3} - 3\log\left|1+(1-x)^{1/3}\right| + c.$
51. $\frac{1}{5}e^x(\cos 2x + 2\sin 2x) + c.$
52. $\frac{1}{3}x^3\log x - \frac{1}{9}x^3 + c.$
53. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c.$
54. $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$
55. $\frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x + c.$
56. $\frac{1}{2}\sec x\operatorname{tg} x + \frac{1}{2}\log|\sec x + \operatorname{tg} x| + c.$
57. $\operatorname{tg} x + \sec x + c.$
58. $\frac{-1}{2}x\cos(\log x) + \frac{1}{2}x\sin(\log x) + c.$
59. $-\frac{\sqrt{1-x^2}}{x} + c.$
60. $\sqrt{1+x^2} + c.$
61. $\operatorname{arc\,tg}\sqrt{e^{2x}-1} + c.$
62. $\frac{1}{2}e^{2x} - 2e^x + \log\left(e^{2x}+2e^x+2\right) + 2\operatorname{arc\,tg}(e^x+1) + c.$
63. $\frac{1}{2}x^2 + \frac{1}{4(x-1)} - \frac{1}{4(x+1)} + c.$
64. $\frac{-3}{2}(1-2x)^{1/3} - 3(1-2x)^{1/6} - 3\log|(1-2x)^{1/6}-1| + c.$
65. $-\frac{\sqrt{9-x^2}}{9x} + c.$
66. $\frac{-1}{3(x-1)} - \frac{1}{3}\log|x-1| + \frac{1}{6}\log(x^2+x+1) + \frac{1}{3\sqrt{3}}\operatorname{arc\,tg}\frac{2x+1}{\sqrt{3}} + c.$

67. $\frac{1}{m+1}\,x^{m+1}\log x - \frac{1}{(m+1)^2}\,x^{m+1} + c$, if $m \neq -1$; $\frac{1}{2}\,(\log x)^2$ if $m = -1$.
68. $\frac{-1}{3}\,\csc^3 x + \csc x + c$.
69. $\frac{2}{3}\,(-x^{3/2}\cos x^{3/2} + \sin x^{3/2}) + c$.
70. $\frac{1}{2}\,x + \frac{1}{10}\,x\cos(2\log x) + \frac{1}{5}\,x\sin(2\log x) + c$.
71. $x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x + c$.
72. $\frac{1}{2}\,x^2(\log x)^2 - \frac{1}{2}\,x^2\log x + \frac{1}{4}\,x^2 + c$.

**Problem 1.2** Find recurrence relations for the following antiderivatives:

$$i)\quad H_n = \int \log^n x\,dx, \qquad ii)\quad I_n = \int \sin^n x\,dx,$$

$$iii)\quad J_n = \int x^n e^{-x}\,dx, \qquad iv)\quad K_n = \int \frac{dx}{(x^2+1)^n},$$

$$v)\quad L_n = \int \mathrm{tg}^n x\,dx, \qquad vi)\quad M_n = \int \sec^n x\,dx,$$

$$vii)\quad N_n = \int x^n e^{ax}\,dx, \qquad viii)\quad P_n = \int x^n e^{x^2}\,dx.$$

*Solution:* $i)$ $H_n = x\log^n x - nH_{n-1}$; $ii)$ $I_n = \frac{-1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}I_{n-2}$; $iii)$ $J_n = -x^n e^{-x} + nJ_{n-1}$; $iv)$ $K_n = \frac{1}{2(n-1)}\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}K_{n-1}$; $v)$ $L_n = \frac{1}{n-1}\,\mathrm{tg}^{n-1}x - L_{n-2}$. $vi)$ $M_n = \frac{1}{n-1}\,\mathrm{tg}\,x\sec^{n-2}x + \frac{n-2}{n-1}M_{n-2}$; $vii)$ $N_n = \frac{1}{a}x^n e^{ax} - \frac{n}{a}N_{n-1}$; $viii)$ $P_n = \frac{1}{2}x^{n-1}e^{x^2} - \frac{n-1}{2}P_{n-2}$.

**Problem 1.3**

I) Compute $\displaystyle\int_a^b x\,dx$ using upper and lower sums associated to regular partitions of the interval $[a,b]$.

II) Do it also for $\displaystyle\int_a^b x^2\,dx$.

Hint: $i)$ $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$; $ii)$ $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

**Problem 1.4** Prove and interpret the following identities:

$$i)\qquad \int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$$

$$ii)\qquad \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$iii)\qquad \int_{-a}^{a} \Big(f(x) - f(-x)\Big)\,dx = 0$$

$$iv)\qquad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

$$v)\qquad \int_1^a \frac{dx}{x} + \int_1^b \frac{dx}{x} = \int_1^{ab} \frac{dx}{x}.$$

**Problem 1.5**

I) Prove that, if $g$ is an odd and integrable function on $[-a, a]$, then $\int_{-a}^{a} g = 0$. Apply the result to compute

$$\int_6^{10} \sin[\sin\{(x-8)^3\}]dx\,.$$

II) Prove that, if $h$ is an even and integrable function on $[-a, a]$, then $\int_{-a}^{a} h = 2\int_0^a h$.

**Problem 1.6** Let $g$ be a continuous function and $g'$ continuous on $[a, b]$, such that $g(a) = g(b)$ and $\alpha \le g(x) \le \beta$ for all $x \in [a, b]$. Show that for every integrable $f$ on $[\alpha, \beta]$,

$$\int_a^b f(g(x))g'(x)\,dx = 0.$$

Is it true that $\int_a^b f(g(x))\,dx = 0$?
Hint: Change variables.

**Problem 1.7** Let $f,\, g : [-1, 1] \to I\!R$ be piecewise continuous functions.

I) Show that

$$\int_{-\pi}^{0} f(\sin x)\; g(\cos x)\,dx = \int_{\pi}^{2\pi} f(\sin x)\; g(\cos x)\,dx.$$

II) Let $f$ be an odd function. Prove that

$$\int_0^{2\pi} f(\sin x)\,g(\cos x)\,dx = 0.$$

III) Indicate which of the following integrals are equal to zero:

$a)\quad \int_0^{2\pi} \mathrm{e}^{|\cos x|}\,\mathrm{arc\,tg}(\sin^5 x)\,dx,\qquad b)\quad \int_0^{2\pi} \mathrm{e}^{\cos x}\mathrm{e}^{-|\sin x|}\,dx,$

$c)\quad \int_0^{2\pi} \frac{\sin(\cos^4 x)\sin(\sin x)}{(1+\cos^2 x)(3-\sin^2 x)}\,dx.$

**Problem 1.8** Let $f$ be a periodic function of period $T$, integrable on $[0, T]$.

I) Prove that for all $a \in I\!R$, we have

$$\int_a^{a+T} f = \int_0^T f.$$

II) Prove that for all integer $n$

$$\int_a^b f = \int_{a+nT}^{b+nT} f.$$

Hint: $i)$ Separate the integral on $[a, a+T]$ as the sum of integrals on $[a, T]$ and $[T, a+T]$, and use a change of variables in the second integral;
$ii)$ Change variables.

**Problem 1.9** Using the two previous problems, prove that if $f$ and $g$ are piecewise continuous functions on $[-1,1]$, and $g$ is a an odd function, we have

$$\int_0^{2\pi} f(\sin x)\, g(\cos x)\, dx = 0.$$

Hint: Apply the change of variables $t = \pi/2 - x$, and use periodicity.

**Problem 1.10**

I) Prove that if $f$, $g$ are piecewise continuous functions on $[-1,1]$ and $n > 0$

$$\int_a^{a+2\pi/n} f(\sin nx)\, g(\cos nx)\, dx = 0,$$

being odd at less one of the functions $f$ or $g$.

II) What is the value of the same integral on the interval $[a, a+2\pi]$ if $n \in \mathbb{N}$?

III) Indicate which of the following integrals are equal to zero:

$$a)\quad \int_\pi^{2\pi} \cos(\cos 2x)\, \operatorname{arc\,tg}(\sin^5 2x)\, dx, \qquad b)\quad \int_{\pi/2}^{5\pi/2} \operatorname{tg}(\cos x)\, dx$$

$$c)\quad \int_{-\pi}^{3\pi} \frac{\sin(\cos^4 x)\sin(\sin x)}{(1+\cos^2 x)(3-\sin^2 x)}\, dx \qquad d)\quad \int_0^{2\pi} \operatorname{tg}\left(\frac{1+\cos x}{2}\right) dx.$$

IV) Prove that if $f$ and $g$ are even functions, then

$$\int_0^{2\pi} f(\sin x)\, g(\cos x)\, dx = 4\int_0^{\pi/2} f(\sin x)\, g(\cos x)\, dx.$$

**Problem 1.11** Evaluate the following limits associating them to some definite integral:

$$i)\quad \lim_{n\to\infty} \left[\frac{n}{n^2+1} + \frac{n}{n^2+4} + \cdots + \frac{n}{n^2+n^2}\right]$$

$$ii)\quad \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right]$$

$$iii)\quad \lim_{n\to\infty} \frac{\sqrt[n]{e^2} + \sqrt[n]{e^4} + \cdots + \sqrt[n]{e^{2n}}}{n}$$

$$iv)\quad \lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$v)\quad \lim_{n\to\infty} \left[\frac{1}{n^2+1} + \frac{1}{n^2+4} + \cdots + \frac{1}{n^2+n^2}\right].$$

*Solution:* $i)$ $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$; $ii)$ $\int_0^1 \frac{dx}{1+x} = \log 2$; $iii)$ $\int_0^1 e^{2x} dx = (e^2-1)/2$; $iv)$ $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$; $v)$ $\int_0^1 \frac{dx}{1+x^2} \cdot \lim_{n\to\infty} \frac{1}{n} = 0$; (also $0 \le \sum_{j=1}^{n} \frac{1}{n^2+j^2} \le \frac{n}{n^2+1} \to 0$).

**Problem 1.12** Compute the limit

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k}{n}\right)^{1/n}.$$

*Solution:* $e^{\int_0^1 \log(1+x)dx} = 4/e$.

**Problem 1.13** Evaluate $F(x) = \int_{-1}^{x} f(t)\,dt$ with $x \in [-1,1]$, for the following functions:

*i)* $f(x) = |x-1/2|$

*ii)* $f(x) = |x|\,e^{-|x|}$

*iii)* $f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x \le 1 \end{cases}$

*iv)* $f(x) = \begin{cases} x^2 & -1 \le x < 0 \\ x^2-1 & 0 \le x \le 1 \end{cases}$

*v)* $f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ x+1 & 0 < x \le 1 \end{cases}$

*vi)* $f(x) = \begin{cases} x+2 & -2 \le x \le -1 \\ 1 & -1 < x < 1 \\ -x+2 & 1 \le x \le 2 \end{cases}$

*vii)* $f(x) = \max\{\sin(\pi x/2), \cos(\pi x/2)\}$.

*Solution:* *i)* $F(x) = \begin{cases} -x^2/2 + x/2 + 1 & \text{if } -1 \le x \le 1/2 \\ x^2/2 - x/2 + 5/4 & \text{if } 1/2 \le x \le 1 \end{cases}$;
*ii)* $F(x) = \begin{cases} (1-x)e^{x} - 2e^{-1} & \text{if } x \le 0 \\ (-1-x)e^{-x} + 2 - 2e^{-1} & \text{if } x \ge 0 \end{cases}$; *iii)* $F(x) = |x| - 1$;
*iv)* $F(x) = \begin{cases} (x^3+1)/3 & \text{if } -1 \le x \le 0 \\ -x + (x^3+1)/3 & \text{if } 0 \le x \le 1 \end{cases}$;
*v)* $F(x) = \begin{cases} 1+x & \text{if } -1 \le x \le 0 \\ 1+x+x^2/2 & \text{if } 0 \le x \le 1 \end{cases}$; *vi)* $x+1$.

**Problem 1.14** Let $F(x) = \int_a^x f(t)\,dt$ with $f$ integrable.

I) Prove that if $|f| \le M$ on the interval $[\alpha,\beta]$, then $|F(x) - F(y)| \le M|x-y|$ for all $x, y \in [\alpha,\beta]$, implying the (uniformly) continuity of $F$.

II) Is $F$ differentiable necessarily? Under what conditions can we say that is differentiable?

Hint: *i)* $F(x) - F(y) = \int_y^x f(t)dt$.

**Problem 1.15** Differentiate the following functions:

*i)* $F(x) = \int_{x^2}^{x^3} \frac{e^t}{t}\,dt$

*ii)* $F(x) = \int_{-x^3}^{x^3} \frac{dt}{1+\sin^2 t}$

*iii)* $F(x) = \int_{3}^{\int_1^x \sin^3 t\,dt} \frac{ds}{1+\sin^6 s + s^2}$

*iv)* $F(x) = \int_{2}^{e^{\int_1^{x^2} \operatorname{tg}\sqrt{t}\,dt}} \frac{ds}{\log s}$

*v)* $F(x) = \int_0^x x^2 f(t)dt$, with $f$ continuous on $I\!R$,

*vi)* $F(x) = \sin\left(\int_0^x \sin\left(\int_0^y \sin^3 t dt\right) dy\right)$.

*Solution:* *i)* $3e^{x^3}/x - 2e^{x^2}/x$; *ii)* $6x^2/(1+\sin^2 x^3)$;
*iii)*$(\sin x)^3/[1+\sin^6(\int_1^x \sin^3 t dt) + (\int_1^x \sin^3 t dt)^2]$; *iv)* $2x \operatorname{tg}|x| e^{\int_1^{x^2} \operatorname{tg}\sqrt{t}\, dt} / (\int_1^{x^2} \operatorname{tg}\sqrt{t}\, dt)$;
*v)* $2x\int_0^x f(t)dt + x^2 f(x)$; *vi)* $\cos(\int_0^x \sin(\int_0^y \sin^3 t dt)dy)\ \sin(\int_0^x \sin^3 t dt)$.

**Problem 1.16** Find the points were $f$ attains its absolute maximum and minimum on $[1,\infty)$, where $f$ is:

$$f(x) = \int_0^{x-1} \left(e^{-t^2} - e^{-2t}\right) dt\,,$$

knowing that $\lim_{x\to\infty} f(x) = (\sqrt{\pi}-1)/2$. Is the absolute minimum of $f$ on $[1,\infty)$ also its absolute minimum on $\mathbb{R}$?

*Solution:* $x=1$ minimum, $x=3$ maximum. Yes.

**Problem 1.17** Find the tangent line to the graph of $f(x) = \displaystyle\int_{x^2}^{\sqrt{\pi}/2} \operatorname{tg}(t^2)\, dt$ at the point $x = \sqrt[4]{\pi/4}$.

*Solution:* $y = -\sqrt[4]{4\pi}x + \sqrt{\pi}$.

**Problem 1.18** Compute the following limits:

$$i)\quad \lim_{x\to 0} \frac{\displaystyle\int_0^x e^{t^2}\, dt - x}{x^3} \qquad ii)\quad \lim_{x\to 0} \frac{\cos x \displaystyle\int_0^x \sin t^3\, dt}{x^4}.$$

*Solution:* *i)* 1/3; *ii)* 1/4.

**Problem 1.19** If the integral $\displaystyle\int_{-1/x}^{x} \frac{dt}{a^2+t^2}$ does not depend on $x$, find $a$ without computing the integral.

*Solution:* $a = \pm 1$.

**Problem 1.20** Find the continuous function $f$ verifying $xf(x) = \displaystyle\int_0^x f(t)\, dt,\ \ f(0)=1$.

*Solution:* $f(x) = 1$.

**Problem 1.21** If the function $g$ is given by the equation

$$t = \int_0^{|g(t)|^2} \frac{\sin x}{x}\, dx,$$

compute:

I) $g'(t)$ in terms of $g(t)$,
II) $(g^{-1})'(x)$.

*Solution:* *i)* $g'(t) = \frac{g(t)}{2\sin(g(t)^2)}$; *ii)* $(g^{-1})'(x) = \frac{2\sin x^2}{x}$.

**Problem 1.22** The equation

$$\int_1^{f(x)} e^{-t^2}\,dt - 2x + \log(\cos x) = 0$$

defines a differentiable and one to one function $f$ on the interval $[-1/2, 1/2]$. Find:

I) $f(0),\ f'(0)$ and $(f^{-1})'(1)$,

II) $\lim_{x\to 0} \dfrac{e^x - e^{-\sin x}}{f^{-1}(x+1)}$.

*Solution:* *i)* $f(0) = 1,\ f'(0) = 2e,\ (f^{-1})'(1) = 1/(2e)$; *ii)* 4e.

**Problem 1.23** The equation

$$\int_0^{g(x)} (e^{t^2} + e^{-t^2})\,dt - x^3 - 3\,\mathrm{arc\,tg}\,x = 0$$

defines a differentiable and one to one function $g$ on $\mathbb{R}$. Find:

I) $g(0),\ g'(0)$ and $(g^{-1})'(0)$,

II) $\lim_{x\to 0} \dfrac{g^{-1}(x)}{g(x)}$.

*Solution:* *i)* $g(0) = 0,\ g'(0) = 3/2,\ (g^{-1})'(0) = 2/3$; *ii)* 4/9.

**Problem 1.24**

I) Find the explicit formula of a function $f : \mathbb{R} \to \mathbb{R}$ verifying

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C.$$

Next, find the value of $C$.

II) Do the same problem for the function $g$:

$$\int_0^x g(t)\,dt = \int_x^1 g(t)\sin t\,dt - \cos x + D.$$

III) Find $\lim_{x\to 0} \dfrac{\int_0^{x^2} e^{-t^2}\,dt}{\int_0^x g(t)dt}$.

*Solution:* *i)* $f(x) = 2x^{15}$; $C = -1/9$; *ii)* $g(x) = \frac{\sin x}{1+\sin x}$ and $D = \cos 1 + \sec 1 - \mathrm{tg}\,1$; *iii)* 2.

**Problem 1.25** Let $f$ be a continuous and strictly positive function on the interval $[0, 1]$. Define the following function

$$F(x) = 2\int_0^x f(t)\,dt - \int_x^1 f(t)\,dt.$$

Prove that $F$ annihilates on the interval $(0, 1)$ and it happens only once .

Hint: Show that $F$ is monotonic and changes sign.

**Problem 1.26** Evaluate the following definite integrals, changing the limits of integration when making a change of variables:

$$i)\quad \int_0^{\log 2} \sqrt{e^x - 1}\,dx \qquad ii)\quad \int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx.$$

*Solution:* $i)$ $t = \sqrt{e^x - 1}$; the integral is $2 - \pi/2$; $ii)$ $t = \sqrt{x^2-1}$ (or $x = \sec t$); the integral is $\sqrt{3} - \pi/3$.

**Problem 1.27** Consider the functions

$$f(x) = \int_0^x e^{t^2 - x^2}\,dt, \qquad g(x) = \int_0^x e^{t^2}\,dt, \qquad h(x) = \frac{e^{x^2}}{2x},$$

I) Evaluate $\lim\limits_{x\to\infty} \dfrac{g(x)}{h(x)}$;

II) Express $f$ in terms of $g$ and $h$, and express $f'$ in terms of $f$;

III) Prove, using the previous parts, that $\lim\limits_{x\to\infty} x f(x) = 1/2$ and $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} f'(x) = 0$.

*Solution:* $i)$ 1; $ii)$ $f(x) = g(x)/(2xh(x))$, $f'(x) = -2xf(x) + 1$.

**Problem 1.28** Evaluate the following limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n^2 - 1}{(x^2+1)(1+n^2)} e^{-x^4/n}\,dx.$$

*Solution:* $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi$.

**Problem 1.29** Analyze the convergence of the following improper integrals:

$$i)\ \int_{-\infty}^{\infty} e^{-x}\,dx \qquad ii)\ \int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad iii)\ \int_1^{\infty} e^{-x} x^p\,dx$$

$$iv)\ \int_0^a e^{1/x} x^p\,dx \qquad v)\ \int_0^1 \log x\,dx \qquad vi)\ \int_1^{\infty} \frac{dx}{x^\alpha \sqrt{1+x^2}}$$

$$vii)\ \int_0^1 x^p (1-x)^q\,dx \qquad viii)\ \int_1^{\infty} \left(\frac{1}{\sqrt{x}} - \operatorname{arc\,tg} \frac{1}{\sqrt{x}}\right) dx \qquad ix)\ \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+4}\,dx$$

$$x)\ \int_1^2 \frac{\log t + t - 1}{(t-1)^{3/2}}\,dt \qquad xi)\ \int_0^{1/2} \frac{e^{\arcsin x}(5 - \sin x)}{x(\log x)^2 (5 + \sin x)}\,dx \qquad xii)\ \int_0^{\infty} \frac{dx}{x^2 + a^2}.$$

*Solution:* $i)$ D; $ii)$ C; $iii)$ C $\forall p$; $iv)$ D $\forall p$, if $a > 0$; C $\forall p$, if $a \le 0$; $v)$ C; $vi)$ C $\forall \alpha > 0$; $vii)$ C $\forall p, q > -1$; $viii)$ C; $ix)$ C $\forall a \in \mathbb{R}$; $x)$ C; $xi)$ C; $xii)$ C $\forall a \neq 0$.

**Problem 1.30**

i) Analyze the convergence of the integral

$$\int_0^1 \log x \, \log(x+1)\, dx.$$

ii) Compute its value knowing that $\displaystyle\int_0^1 \frac{\log(x+1)}{x}\, dx = \pi^2/12$.

iii) Prove (without calculator!) that $2\log 2 + \pi^2/12 > 2$.

*Solution:* *i*) The integral converges; *ii*) and is equal to $2 - 2\log 2 - \pi^2/12$.

**Problem 1.31** Using the value of the integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$, prove that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \alpha t}{t}\, dt = \begin{cases} 1 & \text{if} \quad \alpha > 0 \\ 0 & \text{if} \quad \alpha = 0 \\ -1 & \text{if} \quad \alpha < 0 \end{cases}.$$

With this result and the formula

$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b$$

prove also that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin t \, \cos xt}{t}\, dt = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 1/2 & \text{if} \quad |x| = 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}.$$

Hint: It is enough to prove it for $x \geq 0$, because the integrand and the solution are even functions on $x$.

**Problem 1.32** Let the integral $\displaystyle I_k = \int_{-\infty}^\infty x^k e^{-x^2}\, dx$.

i) Prove that $I_k$ is convergent for all $k \in \mathbb{N}$.

ii) Evaluate the integral if $k$ is odd.

iii) Find a recurrence relation for $I_{2n}$.

iv) Use the previous formula to show, knowing that $I_0 = \sqrt{\pi}$, that

$$I_{2n} = \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right)\cdots\frac{3}{2}\,\frac{1}{2}\,\sqrt{\pi}.$$

*Solution:* *ii*) $I_{2n+1} = 0$; *iii*) $I_{2n} = (n - 1/2)I_{2n-2}$.

**Problem 1.33** Given the integral

$$I_p = \int_c^1 \frac{t^p}{\sqrt[4]{t^3(1-t)}}\,dt, \qquad 0 \le c < 1.$$

I) Analyze its convergence.

II) Deduce a relation between $I_p$ and $I_{p-1}$.

Hint: *ii*) Do integration by parts, differentiating $t^{p-3/4}$, to obtain

$$I_p = \frac{4}{3}c^{p-3/4}(1-c)^{3/4} + \Big(\frac{4p}{3} - 1\Big)(I_{p-1} - I_p).$$

*Solution:* *i*) If $c = 0$, $I_p$ converges for $p > -1/4$; if $c > 0$ converges always; *ii*) the relation is

$$I_p = \frac{1}{p}c^{p-3/4}(1-c)^{3/4} + \Big(1 - \frac{3}{4p}\Big)I_{p-1}, \qquad p > 3/4.$$

**Problem 1.34** Prove that the following improper integrals converge only for the given values of the parameter:

I) $\displaystyle\int_0^{\pi/2} \log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{dx}{\cos x}$, with $|s| \le 1$.

II) $\displaystyle\int_0^{\infty} \log\left(1+\frac{a^2}{x^2}\right)dx$, with $a \in I\!R$.

III) $\displaystyle\int_0^{1} \frac{x^p-1}{\log x}dx$, with $p > -1$.

**Problem 1.35** Consider the improper integral

$$\int_0^\infty \Big(\frac{1}{\sqrt{1+x^2}} - \frac{\alpha}{x+1}\Big)\,dx, \qquad \alpha > 0.$$

I) Prove that this integral only converges for a value of the parameter $\alpha$ and find that value.

II) Find an antiderivative of the function $(1+x^2)^{-1/2}$.

III) Evaluate the improper integral for the computed value of $\alpha$.

*Solution:* *i*) $\alpha = 1$; *ii*) $\log(x+\sqrt{1+x^2})$; *iii*) $\log 2$.

**Problem 1.36** If we define the following function for $x > 0$

$$f(x) = \int_{1/x}^{x^2} t^\alpha \mathrm{e}^{-t^2}\,dt.$$

I) Find the tangent line to the graph of $f$ at $x = 1$.

II) Find the horizontal and vertical asymptotes of $f$ for the different values of $\alpha$.

*Solution:* *i*) $y = 3(x-1)/\mathrm{e}$; *ii*) if $\alpha > -1$, there is HA, $y = \int_0^\infty t^\alpha \mathrm{e}^{-t^2}dt$, there is not VA; if $\alpha \le -1$, there is not HA, there is VA, $x = 0$.

**Problem 1.37** Let $f$ be continuous on $[0,1]$, compute the limit

$$\lim_{x\to 0} x \int_{x^2}^{1} \frac{f(t)}{t}\, dt.$$

Hint: If the integral converges, the limit is zero; if it diverges, apply L'Hôpital. Alternative form: if $f = 1$, the integral is explicit; if $f$ is continuous on $[0,1]$, $f$ is bounded.

*Solution:* 0.

**Problem 1.38** Let $f$ be a continuous function such that $\lim_{x\to\infty} f(x) = l$,

I) Prove that for all $r > 0$ fixed, we have $\lim_{x\to\infty} \frac{1}{2r} \int_{x-r}^{x+r} f(t)\, dt = l$.

II) If we also have $l > 0$ and $\sum_{k=0}^{n} f(k) \neq 0$ for all $n \in \mathbb{N}$, compute $\lim_{n\to\infty} \frac{\int_0^n f(t)\, dt}{\sum_{k=0}^{n} f(k)}$.

Hint: *ii*) Use Stolz test and part *i*).

*Solution:* *ii*) 1.

**Problem 1.39** Obtain the power series of $f(x) = \text{arctg}x$, integrating term by term the power series of $f'(x)$.

*Solution:* $\text{arc tg}\, x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1 \leq x \leq 1$.

**Problem 1.40** Prove the following identities:

I) $\lim_{b\to+\infty} \int_0^{\infty} e^{-ax} \sin bx\, dx = 0 \qquad (a > 0)$,

II) $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi$,

III) $\int_0^{\infty} t^n e^{-t} dt = n! \qquad (n \in \mathbb{N})$.

Hint: *i*) Compute the integral using parts twice; *ii*) use a change of variables transforming the interval $[a,b]$ onto $[-1,1]$; *iii*) use a recurrence relation already proven in a previous problem.

**Problem 1.41** Consider isosceles triangles with base the line segments $(n, n+1/2^n)$ and height 1, where $n \in \mathbb{N}$ $(n \geq 0)$. Let $f$ be the function whose graph is the polygonal made by the x axis and the triangles. Show that $\int_0^{\infty} f = 1$.

**Problem 1.42** Given the function $f(x) = \begin{cases} (x-1)^2 & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$, define the sequence given by $f_n(x) = nf(nx)$. Evaluate

$$i) \quad \lim_{n\to\infty} \int_0^1 f_n(x)\, dx, \qquad ii) \quad \int_0^1 \lim_{n\to\infty} f_n(x)\, dx.$$

Explain what is happening.

*Solution:* *i*) 1/3; *ii*) 0.

**Problem 1.43**

I) Analyze the convergence of the improper integral $\int_1^\infty \frac{e^x}{x^x}\,dx$ by comparing with the corresponding series.

II) Analyze the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ by comparing with the corresponding integral. Apply a change of variables to this integral to transform it into the previous one.

III) Analyze the convergence of the series

$$a)\quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}} \quad \alpha > 0, \qquad b)\quad \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log(\log n)}}.$$

Hint: *iii.b*) Make a change of variables and use the identity $t^\alpha = e^{\alpha \log t}$, for $\alpha, t > 0$.

*Solution: i*) C, use root test ; *iii.a*) C, $\forall\, \alpha > 1$; *iii.b*) D, because $\lim_{t\to\infty} e^{t-\log^2 t} = \infty$.

**Problem 1.44** To study the convergence of the integral $\int_0^\infty (-1)^{[x^2]}\,dx$, consider the limit

$$\lim_{R\to\infty} \int_0^R (-1)^{[x^2]}\,dx = \lim_{R\to\infty} \Big( \int_0^{\sqrt{[R^2]}} (-1)^{[x^2]}\,dx + \int_{\sqrt{[R^2]}}^R (-1)^{[x^2]}\,dx \Big),$$

and analyze each limit separately.

I) Show that $\lim_{R\to\infty} (R - \sqrt{[R^2]}) = 0$.

II) Let $M \in \mathbb{N}$, use the change of variables $x^2 = t$ to prove the formula

$$\int_0^{\sqrt{M}} (-1)^{[x^2]}\,dx = \sum_{n=1}^{M} (-1)^{n-1} \Big( \sqrt{n} - \sqrt{n-1} \Big).$$

III) Conclude the convergence of the integral.

IV) Is it absolutely convergent?

**Problem 1.45**

I) Evaluate $\int_1^n \log x\,dx$. Compare the previous integral with its upper and lower sums for the partition $P = \{1, 2, \ldots, n\}$. Deduce from the previous comparison the following inequality:

$$(n-1)! \le n^n e^{-n+1} \le n!\,.$$

II) Evaluate $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$.

*Solution: ii*) $1/e$.

**Problem 1.46** Find the area enclosed by the following curves:

*i)* $y = x^2,\ y = (x-2)^2,\ y = (2-x)/6,$

*ii)* $x^2 + y^2 = 1,\ x^2 + y^2 = 2x,$

*iii)* $y = (1-x)/(1+x),\ y = (2-x)/(1+x),\ y = 0,\ y = 1,$

*iv)* loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

*Solution:* *i)* 25/48; *ii)* $2\pi/3 - \sqrt{3}/2$; *iii)* $\log 2$; *iv)* $8(b-a)^{5/2}/15$.

**Problem 1.47** Find the area enclosed by the following curves given in parametric and polar coordinates:

*i)* loop: $x = t^2 + 1,\ y = t(t^2 - 4), \quad -2 \le t \le 2,$

*ii)* cycloid: $x = a(t - \sin t),\ y = a(1 - \cos t), \quad 0 \le t \le 2\pi$, and axis $x$,

*iii)* spiral of Archimedes: $r = a\theta, \quad 0 \le \theta \le 2\pi$, and axis $x$,

*iv)* three-leaved rose: $r = a\cos 3\theta$,

*v)* lemniscate: $r = a\sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4$.

*Solution:* *i)* 256/15; *ii)* $3\pi a^2$; *iii)* $4a^2\pi^3/3$; *iv)* $\pi a^2/4$; *v)* $a^2/4$.

**Problem 1.48** Find the area between the graph of the function $f(x) = \dfrac{x^2-4}{x^2+4}$ and its asymptote.

*Solution:* The asymptote is $y = 1$ and the area $4\pi$.

**Problem 1.49** Let $A$ be the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$. Compute the area of $A$ and the revolution volume obtained by rotating $A$ about the horizontal axis.

*Solution:* The area is 1/3 and the volume $3\pi/10$.

**Problem 1.50** Evaluate the volumes formed by revolving the following regions about the $x$ axis:

*i)* $0 \le y \le 1 + \sin x, \quad 0 \le x \le 2\pi,$

*ii)* $x^2 + (y-2a)^2 \le a^2$, the graph is the torus,

*iii)* $R^2 \le x^2 + y^2 \le 4R^2$, the graph is an spherical ring,

*iv)* the surface bounded by the curves $y = \sin x$ and $y = x$, with $x \in [0, \pi]$,

*v)* $x = t - \sin t,\ 0 \le y \le 1 - \cos t, \quad 0 \le t \le 2\pi.$

*Solution:* *i)* $3\pi^2$; *ii)* $4\pi^2 a^3$; *iii)* $28\pi R^3/3$; *iv)* $\pi^4/3 - \pi^2/2$; *v)* $5\pi^2$.

**Problem 1.51**

*i)* Compute the volumes formed by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ about the $x$ and $y$ axes.

*ii)* Compute the volume of the solid with base the previous ellipse and whose perpendicular sections to the $x$ axis are isosceles triangles of height 2.

*Solution:* *i*) $4\pi ab^2/3$ and $4\pi a^2 b/3$ respectively; *ii*) $\pi ab$.

**Problem 1.52** Find the length of the following graphs:

I) catenary: $y = \mathrm{e}^{x/2} + \mathrm{e}^{-x/2}, \quad 0 \le x \le 2,$

II) cycloid: $x(t) = a(t - \sin t),\ y(t) = a(1 - \cos t), \quad 0 \le t \le 2\pi,$

III) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4,$

IV) tractrix: $y = a \log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}, \quad a/2 \le x \le a,$

V) cardioid: $r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi,$

VI) circular helix : $x(t) = a\cos t,\ y(t) = a \sin t,\ z(t) = bt, \quad 0 \le t \le 2\pi.$

*Solution:* *i*) $\mathrm{e} - 1/\mathrm{e}$; *ii*) $8a$; *iii*) 48; *iv*) $a \log 2$; *v*) 8; *vi*) $2\pi\sqrt{a^2 + b^2}$.