2. INTEGRATION IN SEVERAL VARIABLES

Problem 2.1 Let $f : Q = [0,1] \times [0,1] \longrightarrow I\!R$ be given by $f(x,y) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$.
Prove that f is integrable and $\int_Q f = \frac{1}{2}$.

Problem 2.2 Let $f : A \subset I\!R^2 \longrightarrow I\!R$ be integrable on A and let $g : A \longrightarrow I\!R$ with $g = f$ except for a finite number of points. Prove that g is integrable on A and $\int_A g = \int_A f$.

Problem 2.3

I) Prove, without computing the integral, that

$$4\pi \le \int_D (x^2 + y^2 + 1)\, dx\, dy \le 20\pi\,,$$

where D is the disk of radius 2 centered at the origin.

II) Let A be the square $[0,2] \times [1,3]$ and let $f(x,y) = x^2 y$. Prove, without computing the integral, that

$$0 \le \int_A f(x,y)\, dx\, dy \le 48\,.$$

III) Improve this last estimation and prove that

$$3 \le \int_A f(x,y)\, dx\, dy \le 25\,.$$

Hint: Use a partition of A consisting of four equal squares.

Problem 2.4 Approximate in the following cases, by means of upper and lower sums, the integral $\int_R f(x,y)\, dA$, where $R = [0,4] \times [0,2]$. Use a partition consisting of eight equal squares. Compute also the integral exactly and compare the results.

i) $f(x,y) = x + y$ *b)* $f(x,y) = xy$
ii) $f(x,y) = x^2 + y^2$ *d)* $f(x,y) = 1/[(x+1)(y+1)]$.

Solution: *i)* $16 < I < 32$, $(I = 24)$; *ii)* $6 < I < 30$, $(I = 16)$; *iii)* $32 < I < 80$, $(I = 160/3)$; *iv)* $77/72 < I < 25/8$, $(I = \log 3 \log 5)$.

Problem 2.5 Let f be the function defined on the square $Q = [0,1] \times [0,1]$:

$$f(x,y) = \begin{cases} 1 - x - y & \text{if } x + y \le 1 \\ 0 & \text{if } x + y \ge 1 \end{cases}$$

Sketch the graph of f over Q and evaluate $\int_Q f$.

Solution: 1/6.

Problem 2.6 *(Cavalieri's Principle)* Let $A, B \subset I\!R^3$ be two regions. If we define the sections $A_c = \{(x,y) \in I\!R^2 \,/\, (x,y,c) \in A\}$ and $B_c = \{(x,y) \in I\!R^2 \,/\, (x,y,c) \in B\}$. Suppose that A_c and B_c have the same area for each value of c. Prove that A and B have the same volume.

Problem 2.7 From the previous problem it follows that two pyramids with the same base and height have the same volume. Find that volume by integrating.

Solution: $V = Ah/3$, where $A =$ area of the base.

Problem 2.8 We call *cone* the three–dimensional picture obtained by joining all the points of a planar region S to a point located out of the S plane. Let A be the area of S and h the height of the cone, show that:

I*)* The area of the section of a parallel plane to the base at a distance t from the vertex is $(t/h)^2 A$, for $0 \le t \le h$.

II*)* The volume of the cone is $Ah/3$.

Problem 2.9 Prove that the following regions of $I\!R^2$ have null measure (area, in this case):

I*)* $S = \{(x,y) \in I\!R^2 \,/\, |x| + |y| = 1\}$,

II*)* $U = \{(x,y) \in I\!R^2 \,/\, x^2 + y^2 = 1\}$,

III*)* the graph of $f : [a,b] \longrightarrow I\!R$, continuous, $G = \{(x, f(x)) \,/\, x \in [a,b]\}$.

Problem 2.10 Let $f : A \subset I\!R^2 \longrightarrow I\!R$ be a function greater than or equal to 0 and integrable on A, with $\displaystyle\int_A f = 0$.

I*)* Let $A_m = \{\mathbf{x} \in A \,/\, f(\mathbf{x}) > 1/m\}$; prove that A_m has null measure (area).

II*)* Deduce that the region where $f(\mathbf{x}) \neq 0$ has null measure.

Are these results true if $A \subset I\!R^n$?

Problem 2.11 Identify the type of region and change the order of integration for the following integrals:

$$i)\ \int_0^3 \int_{4x/3}^{\sqrt{25-x^2}} f(x,y)\,dy\,dx \qquad ii)\ \int_0^1 \int_0^y f(x,y)\,dx\,dy$$

$$iii)\ \int_0^{\pi/2} \int_{-\sin(x/2)}^{\sin(x/2)} f(x,y)\,dy\,dx \qquad iv)\ \int_1^{\mathrm{e}} \int_0^{\log x} f(x,y)\,dy\,dx.$$

Solution: *i)* $\{0 \le y \le 4,\ 0 \le x \le 3y/4\} \cup \{4 \le y \le 5,\ 0 \le x \le \sqrt{25-y^2}\}$; *ii)* $\{0 \le x \le 1,\ x \le y \le 1\}$; *iii)* $\{-\sqrt{2}/2 \le y \le 0,\ -2\arcsin y \le x \le \pi/2\} \cup \{0 \le y \le \sqrt{2}/2,\ 2\arcsin y \le x \le \pi/2\}$; *iv)* $\{0 \le y \le 1,\ \mathrm{e}^y \le x \le \mathrm{e}\}$.

Problem 2.12

I*)* Over the region $R = \{(x,y) \in I\!R^2 \,/\, x^2 + (y-1)^2 \le 1,\ x \ge 0\}$, consider the functions $f(x,y) = \dfrac{1}{\sqrt{1-x^2}}$ and $g(x,y) = \sin(y-1)$. Apply Fubini's Theorem to $\displaystyle\int_R f$ and $\displaystyle\int_R g$ in the two possible ways. Evaluate the integrals for the more convenient order.

II) Find the integral over the same region of the function $h(x,y) = \dfrac{\sqrt{2y^2+x^2}}{y}$.

Solution: *i)* $\int_R f = 2$, $\int_R g = 0$; *ii)* $\int_R h = 1 + \pi/2$.

Problem 2.13 Find the value of the integral $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$.

Solution: 2.

Problem 2.14 Prove the identities

$$i) \quad \int_0^x \int_0^t F(u)\, du dt = \int_0^x (x-u)F(u)\, du$$

$$ii) \quad \int_0^x \int_0^v \int_0^u f(t)\, dt du dv = \frac{1}{2}\int_0^x (x-t)^2 f(t)\, dt.$$

Problem 2.15 Evaluate $\displaystyle\int_D (x^2+y)\, dx\, dy$, where $D = \{ (x,y) \in \mathbb{R}^2 \ : \ |x|+|y| \le 1 \}$.
Hint: Transform the integral into an integral on the region of D in the first quadrant.

Solution: 1/3.

Problem 2.16 Evaluate $\displaystyle\int_0^1 \int_0^1 f(x,y)\, dx\, dy$, where $f(x,y) = \text{máx}(|x|,|y|)$.

Solution: 2/3.

Problem 2.17 Compute the following volumes:

I) volume of intersection of the cylinder $x^2+y^2 \le 4$ and the ball $x^2+y^2+z^2 \le 16$;

II) volume of intersection of the cylinders $x^2+y^2 \le 4$ and $x^2+z^2 \le 4$;

III) volume of the solid bounded by the six cylinders $z^2 = y$, $z^2 = 2y$, $x^2 = z$, $x^2 = 2z$, $y^2 = x$ and $y^2 = 2x$;

IV) volume of the solid bounded by the cones $z = 1 - \sqrt{x^2+y^2}$ and $z = -1 + \sqrt{x^2+y^2}$;

V) volume of the region bounded by the paraboloid $z = x^2+y^2$ and the cylinder $x^2+y^2 = 4$ in $z \ge 0$;

VI) volume of the region bounded by $x^2+y^2+z^2 \le 2$, $x^2+y^2 \le z$ and $z \le 6/5$;

VII) volume of the region bounded by the surfaces $z = x^2+y^2$, $z = 2(x^2+y^2)$, $y = x$ and $y^2 = x$.

Solution: *i)* $32\pi(8-3\sqrt{3})/3$; *ii)* $128/3$; *iii)* $1/7$; *iv)* $2\pi/3$; *v)* 8π; *vi)* $493\pi/750$; *vii)* $3/35$.

Problem 2.18 Let the following mapping be defined by $\left\{ \begin{array}{l} x = u + v \\ y = v - u^2 \end{array} \right.$. Evaluate:

I) the Jacobian $J(u, v)$;

II) the image S on the xy plane of the triangle T on the uv plane of vertices (0,0), (2,0) and (0,2);

III) the area of S;

IV) the integral $\displaystyle\int_S (x - y + 1)^{-2} dxdy$.

Solution: $i)$ $1 + 2u$; $iii)$ $14/3$; $iv)$ $2 + (\pi - 6 \operatorname{arc\,tg}(5/\sqrt{3}))\sqrt{3}/9$.

Problem 2.19 Use a linear mapping to compute $\displaystyle\int_S (x - y)^2 \sin^2(x + y)\, dxdy$, where S is the parallelogram of vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution: $\pi^4/3$.

Problem 2.20 Evaluate $\displaystyle\int_D (y - x)\, dxdy$, where D is the region of the plane bounded by $y = x + 1$, $y = x - 3$, $y = (7 - x)/3$ and $y = 5 - x/3$.

Solution: -8.

Problem 2.21 Find the following areas:

I) area of the region $A = \{(x, y) \in I\!R^2 \ : \ x, y > 0,\ a^2 y \leq x^3 \leq b^2 y,\ p^2 x \leq y^3 \leq q^2 x, \}$, where $0 < a < b$ and $0 < p < q$.

II) area bounded by the curves $xy = 4$, $xy = 8$, $xy^3 = 5$ and $xy^3 = 15$.

Solution: $i)$ $(b - a)(q - p)/2$; $ii)$ $2 \log 3$.

Problem 2.22 Find the integral of the function

$$f(x, y) = \frac{y^4}{b^4 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)} + xy^2$$

over the region $D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$, where a and b are positive constants.

Solution: $3\pi ab(1 - \log 2)/8$.

Problem 2.23 Find the integral of the function

$$f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\, e^{\sqrt{x^2 + y^2}}$$

over the regions $E = \{ x^2 + (y - 1)^2 \leq 1 \}$ and $H = \{ x^2 + (y - 1)^2 \leq 1,\ x \geq 0 \}$.

Solution: 0, 2.

Problem 2.24 Evaluate $\int_D \frac{dx\,dy}{xy}$, where D is the plain region bounded by the curves

$$x^2+y^2=ax\,,\ x^2+y^2=a'x\,,\ x^2+y^2=by\,,\ x^2+y^2=b'y\,,$$

where $0<a<a'\,,\ 0<b<b'$.
Hint: Change variables appropriately, so the new region is the rectangle $[a,a']\times[b,b']$.

Solution: $\log(a'/a)\log(b'/b)$.

Problem 2.25 Evaluate the integral $\int_S \frac{x\,dx\,dy}{4x^2+y^2}$, where S is the region on the first quadrant bounded by the coordinate axes and the ellipses $4x^2+y^2=16,\ 4x^2+y^2=1$.

Solution: 3/4.

Problem 2.26 Let $f(x,y)$ be an odd function on the x variable, that is, $f(-x,y)=-f(x,y)$, and integrable on the region $D\subseteq I\!R^2$ that is symmetric with respect to the x variable (that is, $(x,y)\in D$ if and only if $(-x,y)\in D$). Prove that if f is integrable on D, then $\int_D f=0$.

Problem 2.27 Evaluate the volume of the solid bounded by the surfaces $y=z^2,\ 2y=z^2,\ z=x^2,\ 2z=x^2,\ x=y^2,\ 2x=y^2$.
Hint: Make a change of variables so the new region of integration is the cube $[1,2]^3$. Find the Jacobian of the inverse change.

Solution: 1/7.

Problem 2.28 Let R be the region bounded by the plane $z=3$ and the cone $z=\sqrt{x^2+y^2}$, evaluate

$$i)\quad \int_R \sqrt{x^2+y^2+z^2}\,dxdydz\,,\qquad ii)\quad \int_R \sqrt{9-x^2-y^2}\,dxdydz\,.$$

Solution: *i)* $27\pi(2\sqrt{2}-1)/2$; *ii)* $54\pi-81\pi^2/8$.

Problem 2.29 Evaluate $\int_W f(x,y,z)\,dxdydz$, in the following cases:

I) $f(x,y,z)=\mathrm{e}^{-(x^2+y^2+z^2)^{3/2}}$, and W is the region under the sphere $x^2+y^2+z^2=9$ and over the cone $z=\sqrt{x^2+y^2}$.

II) $f(x,y,z)=z\,\mathrm{e}^{x^2+y^2+z^2}$, and $W=\{\,x^2+y^2\le z^2,\ 0\le z\le 1\,\}$.

III) $f(x,y,z)=\sqrt{1-x^2-y^2}+\frac{xyz^3}{1+z^2}$, and $W=\{\,x^2+y^2+z^2\le 1,\ x^2+y^2\le a^2\,\}$, if $0<a<1$.

Solution: *i)* $\pi(2-\sqrt{2})(1-\mathrm{e}^{-27})/3$; *ii)* $\pi(\mathrm{e}-1)^2/4$; *iii)* $\pi a^2(2-a^2)$.

Problem 2.30 Evaluate the volume of the solid bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Analyze the particular instance $a=b=c=r$.

Solution: *i)* $4\pi abc/3$.

Problem 2.31

I) Evaluate the area of the region $D = \{x = r\cos^3 t,\ y = r\sin^3 t,\ 0 \le r \le 1,\ 0 \le t \le \pi/2\} = \{x^{2/3} + y^{2/3} \le 1,\ x,\ y \ge 0\}$.

II) Find the center of mass of D if its mass density is 1.

Solution: *i)* $3\pi/32$; *ii)* $x_{CM} = y_{CM} = 256/(315\pi)$.

Problem 2.32 The first octant of the ball $x^2+y^2+z^2 \le c^2$ is sliced with the plane $\frac{x}{a} + \frac{y}{b} = 1$, $0 < a, b \le c$. Find the mass of each of the resulting solids knowing that the density is $\rho(x,y,z) = z$.

Solution: One of them is $ab(a^2+b^2-6c^2)/24 + \pi c^2/12$.

Problem 2.33 Find the mass of the sheet corresponding to the portion of the first quadrant of the circle $x^2+y^2 \le 4$, if the density at (x,y) is proportional to the distance between the point and the center of the circle.

Solution: $4\pi\alpha/3$, where α is the proportionality constant.

Problem 2.34 The temperature at points in the cube $[-1,1]^3$ is proportional to the square of its distance from the origin.

I) What is the average temperature?

II) At which points of the cube is the temperature equal to the average temperature?

Solution: *i)* α, where α is the proportionality constant; *ii)* on the unit sphere.

Problem 2.35 Find the center of mass of the hemispherical region of radius R if the density at each point is the square of the distance of the point to the center.

Solution: $(0, 0, 5R/12)$.

Problem 2.36 An ice cream cone is made by a cone of angle α and an ice cream hemisphere of radius R. The cornet and the ice cream have constant densities ρ_c and ρ_h respectively. Determine the value of ρ_c/ρ_h such that the center of mass of the ice cream is located on the plane that separates the ice cream and the cornet.

Solution: $3\,\mathrm{tg}^2\,\alpha$.

Problem 2.37 Evaluate

I) $\displaystyle\int_0^1\int_0^1 \cdots \int_0^1 (x_1^2 + x_2^2 + \cdots + x_n^2)\, dx_1 dx_2 \ldots dx_n$,

II) $\displaystyle\int_0^1\int_0^1 \cdots \int_0^1 (x_1 + x_2 + \cdots + x_n)^2\, dx_1 dx_2 \ldots dx_n$.

Solution: *i)* $n/3$; *ii)* $(3n^2+n)/12$.

Problem 2.38 Let the function

$$I(p,r) = \int_R \frac{dxdy}{(1+x^2+y^2)^p},$$

where R is the disk of radius r centered at the origin. Find the values of p for which $I(p,r)$ has finite limit when $r \to \infty$.

Solution: $\pi p^{2-2p}/(p-1)$ if $p > 1$.

Problem 2.39

I) Evaluate the integral $\displaystyle\int_{D_R} \mathrm{e}^{-(x^2+y^2)} dxdy$, where D_R is the disk of radius R centered at the origin.

II) Let $Q_{a,b}$ be the rectangle $[-a,a] \times [-b,b]$, prove the estimation

$$\int_{D_{r_1}} \mathrm{e}^{-(x^2+y^2)} dxdy \le \int_{Q_{a,b}} \mathrm{e}^{-(x^2+y^2)} dxdy \le \int_{D_{r_2}} \mathrm{e}^{-(x^2+y^2)} dxdy,$$

for certain r_1 and r_2.

III) Taking the limit $a,\, b \to \infty$, prove the formula $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} dx = \sqrt{\pi}$.

Solution: i) $\pi(1-\mathrm{e}^{-R^2})$; *ii)* $r_1 = \text{mín}\{a,\, b\},\ r_2 = \sqrt{a^2+b^2}$.

Problem 2.40 Let f be a continuous function, find $F'(t)$ in the cases

I) $F(t) = \displaystyle\int_0^t \int_0^t \int_0^t f(xyz)\, dxdydz.$

II) $F(t) = \displaystyle\iiint_{x^2+y^2+z^2 \le t^2} f(x^2+y^2+z^2)\, dxdydz,$

Hint: *ii)* Use spherical coordinates.

Solution: i) $3\int_0^t \int_0^t f(txy) dxdy$; *ii)* $4\pi t^2 f(t^2)$.