

3. INTEGRALS DEPENDING ON A PARAMETER

Problem 3.1 Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a continuous function with $\frac{\partial f}{\partial y}$ continuous. Define $F : [a, b] \times [c, d] \rightarrow \mathbb{R}$ as

$$F(x, y) = \int_a^x f(t, y) dt.$$

I) Evaluate $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.

II) Find the derivative of $G(x) = F(g(x), x)$, being g differentiable. Which are the domain and the range of g ?

Solution: ii) $G'(x) = f(g(x), x)g'(x) + \int_a^{g(x)} \frac{\partial f}{\partial y}(t, x) dt$.

Problem 3.2 Find the derivatives of the following functions

$$i) F(y) = \int_{y^2}^y \sin(x^2 + y^2) dx, \quad ii) G(y) = \int_y^{y^2} e^{-x^2 y} dx, \quad iii) H(y) = \int_0^\infty e^{-x^2} \cos(yx) dx.$$

Solution: i) $\sin(2y^2) - 2y \sin(y^4 + y^2) + 2y \int_{y^2}^y \cos(x^2 + y^2) dx$; ii) $2ye^{-y^5} - e^{-y^3} - \int_y^{y^2} x^2 e^{-x^2 y} dx$; iii) $-\int_0^\infty x e^{-x^2} \sin(xy) dx$.

Problem 3.3 Let the function

$$F(x) = \int_0^1 \frac{(\log(1 - xt))^2}{t} dt.$$

I) Find the values of x such that $F(x)$ is defined (that is, the integral converges).

II) Evaluate $F'(x)$ and the resulting integral.

III) Analyze the increasing and decreasing intervals of F .

Solution: i) $(-\infty, 1]$; ii) $F'(x) = (\log(1 - x))^2/x$; iii) F decreases on $(-\infty, 0)$ and increases on $(0, 1)$.

Problem 3.4 Let $F, G : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined as

$$F(x) = \left(\int_0^x e^{-t^2} dt \right)^2 \quad \text{and} \quad G(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt.$$

Prove that:

I) $F'(x) + G'(x) = 0$, for all $x \in \mathbb{R}$.

II) $F(x) + G(x) = \pi/4$, for all $x \in \mathbb{R}$.

III) Deduce that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

Problem 3.5 Evaluate $F(s) = \int_0^\infty e^{-x} \sin(sx) dx$, and, from it, compute

$$G(s) = \int_0^\infty xe^{-x} \cos(sx) dx.$$

Solution: $F(s) = \frac{s}{1+s^2}$, $G(s) = \frac{1-s^2}{(1+s^2)^2}$.

Problem 3.6 Let $F(\alpha) = \int_0^\infty \frac{e^{-\alpha x} - e^{-x}}{x} dx$.

- I) Analyze the convergence of the integral.
- II) Evaluate $F'(\alpha)$ explicitly and, from it, compute $F(\alpha)$.
- III) From the iterated derivatives of $F^{(k)}(\alpha)$, compute

$$\int_0^\infty x^n e^{-x} dx.$$

Solution: i) It converges for $\alpha > 0$; ii) $F'(\alpha) = -1/\alpha$; $F(\alpha) = -\log \alpha$; iii) $n!$.

Problem 3.7 Prove that if h is a differentiable function on $[0, \infty)$, such that $\lim_{x \rightarrow \infty} h(x) = 0$ and if it is possible to differentiate under the integral sign, then, for all $a, b > 0$,

$$\int_0^\infty \frac{h(ax) - h(bx)}{x} dx = h(0) \log(b/a).$$

Problem 3.8 Obtain explicitly the following functions by differentiating and then computing the integral with respect to the parameter:

I) $F(s) = \int_0^{\pi/2} \log\left(\frac{1+s \cos x}{1-s \cos x}\right) \frac{dx}{\cos x}$, with $|s| < 1$.

II) $G(a) = \int_0^\infty \log\left(1 + \frac{a^2}{x^2}\right) dx$, with $a \in \mathbb{R}$.

III) $H(p) = \int_0^1 \frac{x^p - 1}{\log x} dx$, with $p > -1$.

IV) $I(\lambda) = \int_0^{\pi/2} \frac{\log(1 - \lambda^2 \sin^2 x)}{\sin x} dx$, with $|\lambda| < 1$.

V) $K(x) = \int_0^\infty e^{-t^2 - x^2/t^2} dt$, with $x \in \mathbb{R}$.

Hint: ii) As G is an even function, it is enough to consider the case $a \geq 0$; v) make the change of variables $s = x/t$ to prove that $K'(x) = -2K(x)$.

Solution: i) $F(s) = \pi \arcsin s$; ii) $G(a) = \pi|a|$; iii) $H(p) = \log(p+1)$; iv) $I(\lambda) = -(\arcsin \lambda)^2$; v) $K(x) = \sqrt{\pi} e^{-2|x|}/2$.

Problem 3.9 Obtain explicitly the function

$$F(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} dx, \quad \forall t > 0.$$

If F is continuous at 0, deduce the value of the Dirichlet's integral

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Solution: $F(t) = \pi/2 - \operatorname{arctg} t$.

Problem 3.10 Use the identity

$$\int_0^\infty \frac{\cos ax}{1+x^2} dx = \frac{\pi}{2} e^{-a}$$

to prove that

$$\int_0^\infty \frac{\sin ax}{x(1+x^2)} dx = \frac{\pi}{2} (1 - e^{-a}).$$

Problem 3.11 Show that

$$\int_0^{\pi/(4a)} \frac{x}{\cos^2 ax} dx = \frac{1}{2a^2} \left(\frac{\pi}{2} - \log 2 \right).$$

Hint: Differentiate $\operatorname{tg} ax$ with respect to a (or integrate by parts).

Problem 3.12 Prove that

$$J(a) = \int_0^a \frac{dx}{(a^2 + x^2)^2} = \frac{\pi + 2}{8a^3}, \quad \text{if } a > 0.$$

Problem 3.13 Show that

$$\int_0^\pi \frac{\log(1 + \cos x)}{\cos x} dx = \frac{\pi^2}{2},$$

computing first

$$\int_0^\pi \frac{\log(1 + t \cos x)}{\cos x} dx.$$

Solution: The parametric integral is $\pi \cdot \arcsin t$.

Problem 3.14 Prove that

$$\int_0^\infty \frac{1 - e^{-x^2}}{x^2} dx = \sqrt{\pi}.$$

Hint: Do it similarly to the previous problem.

Problem 3.15 Let $F(\lambda) = \int_0^\infty \frac{dx}{x^2 + \lambda}$. Write the derivatives of F and, after computing the integral, prove that for all $\lambda > 0$,

$$\int_0^\infty \frac{dx}{(x^2 + \lambda)^{n+1}} = \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} \frac{\pi}{2\lambda^{n+1/2}} = \frac{(2n)! \pi}{(n!)^2 (2\sqrt{\lambda})^{2n+1}}.$$

Solution: $F(\lambda) = \pi/(2\sqrt{\lambda})$.

Problem 3.16

1) Find, for $\lambda > 0$, the integral

$$\int_0^{\pi/2} (\sin x)^\lambda \cos x dx.$$

II) Use part i) to prove that if $\lambda > 0$ and $n \in \mathbb{N}$, we have

$$\int_0^{\pi/2} (\sin x)^\lambda \cos x [\log(\sin x)]^n dx = \frac{(-1)^n n!}{(1 + \lambda)^{n+1}}.$$

Solution: i) $1/(\lambda + 1)$.

Problem 3.17 Let

$$F(x) = \int_0^{2x} \frac{\log(1 + 2xt)}{1 + t^2} dt, \quad x \geq 0.$$

i) Check that F is differentiable on $(0, \infty)$ and show that

$$F'(x) = \frac{\log(1 + 4x^2)}{1 + 4x^2} + \frac{4x}{1 + 4x^2} \operatorname{arc\,tg} 2x.$$

II) Using part i), prove that

$$F(x) = \log \sqrt{1 + 4x^2} \operatorname{arc\,tg} 2x.$$

Problem 3.18 Suppose that it is possible to differentiate under the integral sign, show that if a and b are positive, then

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b - a).$$

Problem 3.19 Suppose that it is possible to differentiate under the integral sign, show that if a and b are positive, then

$$\int_0^\infty (e^{-a^2/x^2} - e^{-b^2/x^2}) dx = \sqrt{\pi}(b - a).$$

Hint: After computing the partial derivative perform a change of variables.

Problem 3.20

i) Prove that if s is non zero, $t \in \mathbb{R}$, and if we define

$$F(t) = \int_0^\infty e^{-s^2 x^2} \cos(2tx) dx,$$

it is verified the differential equation

$$\frac{F'(t)}{F(t)} = \frac{-2t}{s^2}.$$

II) Obtain $F(t)$ if $F(0) = \frac{\sqrt{\pi}}{2|s|}$.

Solution: $F(t) = \frac{\sqrt{\pi}}{2|s|} e^{-t^2/s^2}$.

Important examples of integrals depending on a parameter and its applications:

Problem 3.21 Prove the following properties of the *gamma* function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0.$$

- I) Γ is continuous and differentiable. Evaluate $\Gamma^{(n)}(x)$.
- II) $\Gamma(1) = \Gamma(2) = 1$; $\Gamma(1/2) = \sqrt{\pi}$.
- III) $\Gamma(x+1) = x\Gamma(x)$.
- IV) Deduce from the previous result that $\lim_{x \rightarrow 0^+} \Gamma(x) = +\infty$.
- V) If $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.
- VI) Find $\Gamma(3/2)$ and $\Gamma(5/2)$.
- VII) If $n \in \mathbb{N}$, $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!} \sqrt{\pi}$.
- VIII) $\int_0^{\infty} e^{-ax} x^n dx = n!/a^{n+1}$, if $a > 0$ and $n \in \mathbb{N}$.

Problem 3.22 If $\int_0^1 \log(\Gamma(x)) dx = K$, compute the value of the integral

$$\int_{\alpha}^{\alpha+1} \log(\Gamma(x)) dx$$

taking first the derivative with respect to the parameter.

Solution: $\alpha(\log \alpha - 1) + K$.

Problem 3.23 Prove the following properties of the *beta* function

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad p, q > 0.$$

- I) $B(p, q) = B(q, p)$.
- II) B is continuous and differentiable on each variable. Prove that

$$\frac{\partial^{n+m} B}{\partial p^n \partial q^m}(p, q) = \int_0^1 x^{p-1} (\log x)^n (1-x)^{q-1} (\log(1-x))^m dx, \quad p, q > 0.$$

III) If $q > 1$, then $B(p, q) = \frac{q-1}{p+q-1} B(p, q-1)$.

IV) If $m, n \in \mathbb{N}$,

$$(m+n+1) B(m+1, n+1) = \binom{m+n}{n}^{-1}.$$

V) $B(p, q) = 2 \int_0^{\pi/2} (\cos t)^{2p-1} (\sin t)^{2q-1} dt$.

VI) $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

$$\text{vii) } B(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt.$$

$$\text{viii) } B(1/2, 1/2) = \pi; \text{ and as a consequence, } \Gamma(1/2) = \sqrt{\pi}.$$

Hint: *vi)* Use the formula $\Gamma(p) = 2 \int_0^\infty x^{2p-1} e^{-x^2} dx$ and change to polar coordinates to compute $\Gamma(p)\Gamma(q)$.

Problem 3.24 Using the *beta* and *gamma* functions, compute the following integrals

$$\begin{array}{ll} i) \int_0^a x^2 \sqrt{a^2 - x^2} dx \quad (a > 0) & ii) \int_0^1 \sqrt{1 - t^2} dt \\ iii) \int_0^1 \log^p(1/x) dx \quad (p > -1) & iv) \int_0^\infty \frac{dx}{(1+x)\sqrt{x}} \\ v) \int_0^\infty x^4 e^{-7x^2} dx & vi) \int_0^\infty \frac{dx}{(7+x)^3 \sqrt{x}}. \end{array}$$

Solution: *i)* $\pi a^4/16$; *ii)* $\pi/4$; *iii)* $\Gamma(p+1)$; *iv)* π ; *v)* $3\sqrt{\pi}/(392\sqrt{7})$.

Problem 3.25

i) Prove the formula for $a, b, c > -1$,

$$\int_D x^a y^b (1-x-y)^c dx dy = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)}{\Gamma(a+b+c+3)},$$

where D is the triangle bounded by the line $x+y=1$ and the coordinate axes.

ii) As an application of part *i)*, prove that, for $p, q, r > 0$,

$$\int_\Omega x^{p-1} y^{q-1} z^{r-1} dx dy dz = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r+1)},$$

with Ω the tetrahedron $\Omega = \{x, y, z \geq 0, x+y+z \leq 1\}$.

iii) Prove the identity

$$\int_V x^{p-1} y^{q-1} z^{r-1} dx dy dz = \frac{a^p b^q c^r}{8} \frac{\Gamma(p/2)\Gamma(q/2)\Gamma(r/2)}{\Gamma((p+q+r+2)/2)},$$

where V is the interior of the ellipsoid in the first octant

$$V = \left\{ x, y, z \geq 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}.$$

iv) Compute the volume of the interior of the ellipsoid.

Hint: *i)* Apply the change of variables $u = x+y, v = \frac{y}{x+y}$, and use the relation between the *beta* and *gamma* functions. *iii)* perform a change of variables transforming ellipsoids into planes.

Solution: *iv)* $4\pi abc/3$.

Problem 3.26 If $f : [0, \infty) \rightarrow \mathbb{R}$ is integrable, and has exponential growth (that is, $|f(t)| \leq ce^{\alpha t}$, for all $t > T$, where c, α, T are certain constants depending on f), the *Laplace transform* of f is defined as

$$L(f)(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

- I) Prove that $L(f)(s)$ converges for $s \in (\alpha, \infty)$ and that is continuous on such interval.
 II) Prove that if $|f(t)| \leq ce^{\alpha t}$, for all $t > 0$, then

$$|L(f)(s)| \leq \frac{c}{s - \alpha}, \quad s > \alpha.$$

Problem 3.27

- I) Prove that if $f(t) \equiv 1$, $L(f)(s) = 1/s$, for $s > 0$.
 II) Making integration by parts, prove that if $f(t) = t^n$, ($n \in \mathbb{N}$), then

$$L(f)(s) = \frac{n!}{s^{n+1}}, \quad s > 0.$$

- III) Using the gamma function, prove that if $f(t) = t^{-1/2}$, then

$$L(f)(s) = \sqrt{\frac{\pi}{s}}.$$

Does this contradict part *ii*) of the previous problem?

Problem 3.28 Prove the following properties of the Laplace transform:

- I) $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$, $\alpha, \beta \in \mathbb{R}$.
 II) Defining $f = 0$ for $t < 0$, then if $a > 0$ we have,

$$L(f(t - a))(s) = e^{-as} L(f)(s).$$

- III) $L(e^{-at} f(t))(s) = L(f)(s + a)$, $a \in \mathbb{R}$.

IV) $L(f(at))(s) = \frac{1}{a} L(f(t))\left(\frac{s}{a}\right)$, $a > 0$.

Problem 3.29 With the previous properties compute the Laplace transform of the following functions, indicating in each case its domain.

- I) $f(x) = e^{ax}$, ($a \in \mathbb{R}$),
 II) $f(x) = x e^{ax}$, ($a \in \mathbb{R}$),
 III) $f(x) = x^n e^{ax}$, ($a \in \mathbb{R}$, $n \in \mathbb{N}$),
 IV) $f(x) = \sin(ax)$, ($a \in \mathbb{R}$),
 V) $f(x) = \cos(ax)$, ($a \in \mathbb{R}$),
 VI) $f(x) = e^{-ax} \cos(bx)$, ($a, b \in \mathbb{R}$),
 VII) $f(x) = e^{-ax} \sin(bx)$, ($a, b \in \mathbb{R}$),
 VIII) $f(x) = \sin^2 x$,

IX) $f(x) = \cos^2 x$.

Solution: *i)* $1/(s-a)$, $s > a$; *ii)* $1/(s-a)^2$, $s > a$; *iii)* $n!/(s-a)^{n+1}$, $s > a$;
iv) $a/(s^2+a^2)$, $s > 0$; *v)* $s/(s^2+a^2)$, $s > 0$; *vi)* $(s+a)/(b^2+(s+a)^2)$, $s > -a$;
vii) $b/(b^2+(s+a)^2)$, $s > -a$; *viii)* $2/[s(s^2+4)]$, $s > 0$; *ix)* $(s^2+2)/[s(s^2+4)]$, $s > 0$.

Problem 3.30 Let f be a continuous function on $[0, \infty)$ with exponential growth.

I) Prove that if f is differentiable on $(0, \infty)$ and f' is continuous, then

$$L(f')(s) = sL(f)(s) - f(0).$$

II) Deduce from *i)* that if f'' is continuous, then

$$L(f'')(s) = s^2 L(f)(s) - s f(0) - f'(0).$$

III) Prove that $L(f)$ is differentiable and verifies

$$\frac{d}{ds}[L(f)(s)] = -L(tf(t))(s).$$

IV) Prove that $L(f)$ has derivatives of all orders, verifying

$$\frac{d^n}{ds^n}[L(f)(s)] = (-1)^n L(t^n f(t))(s).$$

Problem 3.31 Using the previous problem, find the Laplace transform of the following functions, showing in each case its domain:

- I) $f(x) = x^n$ ($n \in \mathbb{N}$),
- II) $f(x) = x e^x$,
- III) $f(x) = x \cos(ax)$ ($a \in \mathbb{R}$),
- IV) $f(x) = x^2 \sin(ax)$ ($a \in \mathbb{R}$),
- V) $f(x) = \sin^3 x$,
- VI) $f(x) = \cos^3 x$.

Hint: *v)* $4 \sin^3 x = 3 \sin x - \sin 3x$; *vi)* $4 \cos^3 x = 3 \cos x + \cos 3x$

Solution: *i)* $n!/s^{n+1}$, $s > 0$; *ii)* $1/(s-1)^2$, $s > 1$; *iii)* $(s^2-a^2)/(s^2+a^2)^2$, $s > 0$;
iv) $(6as^2-2a^3)/(s^2+a^2)^3$, $s > 0$; *v)* $6/[(s^2+1)(s^2+9)]$, $s > 0$;
vi) $(s^3+7s)/[(s^2+1)(s^2+9)]$, $s > 0$.

Problem 3.32 Prove that if f is continuous on $[0, \infty)$ and has exponential growth, then the same is true for the function

$$g(x) = \int_0^x f(t) dt$$

and it is verified

$$L(g)(s) = \frac{1}{s} L(f)(s).$$

Problem 3.33 Use the properties of the Laplace transform to prove that if

$$f(x) = \int_x^\infty \frac{\sin t}{t} dt,$$

then

$$L(f')(s) = \arctan s - \frac{\pi}{2}, \quad L(f)(s) = \frac{\arctan s}{s}.$$

Problem 3.34 Evaluate the Laplace transform of the function

$$f(x) = x \int_0^x e^{-at} \sin(bt) dt, \quad a, b \in \mathbb{R}.$$

Solution: $b(3s^2 + 4as + a^2 + b^2)/(s^2((s+a)^2 + b^2)^2)$.

Problem 3.35

I) Express the Laplace transform of the function $f(x) = x^\alpha$, ($\alpha > -1$), with the help of the gamma function.

II) Find the Laplace transform of the functions

$$a) f(x) = \frac{e^x}{\sqrt{x}}, \quad b) f(x) = x^\alpha e^{ax} \quad (a \in \mathbb{R}, \alpha > -1).$$

Solution: i) $\Gamma(\alpha + 1)/s^{\alpha+1}$; ii.a) $\sqrt{\pi/(s-1)}$; ii.b) $\Gamma(\alpha + 1)/(s-a)^{\alpha+1}$.

Problem 3.36 Find the function whose Laplace transform is

$$\begin{array}{ll} i) \frac{1}{s^2 - 1} & ii) \frac{1}{(s+1)^2} \\ iii) \frac{1}{s(s+1)^2} & iv) \frac{1}{s^n} \quad (n \in \mathbb{N}) \\ v) \frac{1}{(s-1)^2(s^2+1)} & vi) \frac{4s+12}{s^2+8s+16} \\ vii) \frac{s e^{-\pi s/2}}{s^2+a^2} & viii) \frac{1}{\sqrt{s}}. \end{array}$$

Solution: i) $\sinh x = (e^x - e^{-x})/2$; ii) $x e^{-x}$; iii) $1 - (x+1)e^{-x}$; iv) $x^{n-1}/(n-1)!$; v) $\frac{1}{2}((x-1)e^x + \cos x)$; vi) $4(1-x)e^{-4x}$; vii) $\cos(a(x-\pi/2))$ if $x \geq \pi/2$, 0 if $x < \pi/2$; viii) $1/\sqrt{\pi x}$.

Problem 3.37 Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a piecewise continuous function with exponential growth.

I) Prove that if f is periodic of period P , that is, $f(x+P) = f(x)$ for all $x > 0$, then

$$L(f)(s) = \frac{1}{1 - e^{-Ps}} \int_0^P e^{-st} f(t) dt.$$

ii) As an application of the previous formula, compute the transform of Laplace of the function $f(x) = x - [x]$, where $[x]$ denotes the integer part of x .

Hint: i) Divide the integral that defines $L(f)$ in two parts, one defined on $[0, P]$ and the other on $[P, \infty]$. Make an appropriate change of variables in the second integral to exploit the periodicity of f .

Solution: ii) $\frac{1}{s}(\frac{1}{s} - \frac{1}{e^s - 1})$.

Problem 3.38 Solve the following initial value problems

$$\begin{array}{ll}
 i) \quad \begin{cases} y' - 3y = e^{2t} \\ y(0) = 1, \end{cases} & ii) \quad \begin{cases} y' + 3y = \sin 2t \\ y(0) = 0 \end{cases} \\
 iii) \quad \begin{cases} y' - 5y = \cos 3t \\ y(0) = 1/2 \end{cases} & iv) \quad \begin{cases} y'' - y = e^{2t} \\ y(0) = 0, y'(0) = 1 \end{cases} \\
 v) \quad \begin{cases} y'' + 16y = \cos 4t \\ y(0) = 0, y'(0) = 1 \end{cases} & vi) \quad \begin{cases} y'' + 2y' + y = e^{-3t} \\ y(0) = 1, y'(0) = 0 \end{cases} \\
 vii) \quad \begin{cases} y'' - 6y' + 9y = t^2 e^{3t} \\ y(0) = 2, y'(0) = 6 \end{cases} & viii) \quad \begin{cases} y'' + 4y' + 6y = 1 + e^{-t} \\ y(0) = 0, y'(0) = 0. \end{cases}
 \end{array}$$

Solution: i) $y(t) = 2e^{3t} - e^{2t}$; ii) $y(t) = (2e^{-3t} - 2 \cos 2t + 3 \sin 2t)/13$;
 iii) $y(t) = (22e^{5t} - 5 \cos 3t + 3 \sin 3t)/34$; iv) $y(t) = (e^{2t} - e^{-t})/3$;
 v) $y(t) = [(2 + t) \sin 4t]/8$; vi) $y(t) = (e^{-3t} + 3e^{-t} + 6te^{-t})/4$;
 vii) $y(t) = (24 + t^4)e^{3t}/12$; viii) $y(t) = (1 + 2e^{-t} - 3e^{-2t} \cos(\sqrt{2}t) - 2\sqrt{2}e^{-2t} \sin(\sqrt{2}t))/6$.

Problem 3.39 Let f and g be continuous functions on $([0, \infty)$, such that $f(x) = g(x) = 0$ for all $x < 0$, we define the *convolution* of f and g as

$$(f * g)(x) = \int_0^\infty f(y) g(x - y) dy = \int_0^x f(y) g(x - y) dy.$$

- i) Show that the convolution is commutative, that is, $f * g = g * f$.
- ii) Prove that if f or g are differentiable (even though the other is not differentiable), then $f * g$ is differentiable, compute $(f * g)'$.
- iii) Show that if f and g have exponential growth, then its convolution has it also and the following identity is verified

$$L(f * g) = L(f) L(g).$$

- iv) Find $L(f * g)$ if $f(x) = e^x$, $g(x) = \sin x$, and $x \geq 0$, ($= 0$ if $x < 0$).
- v) Find the function whose Laplace transform is

$$a) \frac{1}{(s - 1)(s - 4)} \quad b) \frac{1}{(s^2 + a^2)^2}.$$

- vi) Find the function $f(x)$ verifying the identity

$$f(x) + \int_0^x f(y) e^{x-y} dy - 3x^2 + e^{-x} = 0.$$

Solution: ii) $(f * g)' = f * (g') = (f') * g$; iv) $1/[(s - 1)(s^2 + 1)]$; v.a) $\frac{1}{3}(e^{4t} - e^t)$;
 v.b) $\frac{1}{2a^3}(\sin at - at \cos at)$; vi) $f(x) = 3x^2 - x^3 + 1 - 2e^{-x}$.