4. LINE AND PATH INTEGRALS

Parametrizations of important curves:

Circumference:	$(x-a)^2+(y-b)^2=r^2$	$\Rightarrow$	$\gamma_1(t)=(a+\cos t, b+\sin t)$.
Ellipse:	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\Rightarrow$	$\gamma_2(t)=(a\cos t, b\sin t)$.
Helix:		$\Rightarrow$	$\gamma_3(t)=(\cos t, \sin t, t)$.

Problem 4.1 Sketch the previous curves.

Problem 4.2 Integrate

I) $f(x,y)=2xy^2$ over the first quadrant of the circumference of radius R.

II) $f(x,y,z)=(x^2+y^2+z^2)^2$ along the arc of the circular helix $\mathbf{r}(t)=(\cos t, \sin t, 3t)$, from $(1,0,0)$ to $(1,0,6\pi)$.

Solution: i) $2R^4/3$; *ii)* $2\pi\sqrt{10}(5+120\pi^2+1296\pi^4)/5$.

Problem 4.3 Determine the length and the mass of a thread whose shape is the parabola $y=x^2$ from $(0,0)$ to $(2,4)$ and whose density is $\rho(x,y)=x$.

Solution: The length is $\sqrt{17}+(\log(4+\sqrt{17}))/4$ and the mass is $(17^{3/2}-1)/12$.

Problem 4.4 Evaluate the following integrals, if the closed curves have positive orientation, that is, counterclockwise:

I) $\displaystyle\int_g (x-y)dx+(x+y)dy$, where g is the line segment joining (1,0) to (0,2).

II) $\displaystyle\int_C x^3dy-y^3dx$, where C is the circumference $\{x^2+y^2=1\}$.

III) $\displaystyle\int_\Gamma \frac{dx+dy}{|x|+|y|}$, where Γ is the square of vertices (1,0), (0,1), $(-1,0)$ and $(0,-1)$.

IV) $\displaystyle\int_\rho (x+2y)dx+(3x-y)dy$ where ρ is the ellipse $x^2+4y^2=4$.

V) $\displaystyle\int_R \frac{y^3dx-xy^2dy}{x^5}$, where R is the curve $x=\sqrt{1-t^2}$, $y=t\sqrt{1-t^2}$, $-1\le t\le 1$.

Solution: i) $7/2$; *ii)* $3\pi/2$; *iii)* 0; *iv)* 2π; *v)* $-\pi/2$.

Problem 4.5 Evaluate:

I) $\displaystyle\int_\gamma y\,dx-x\,dy+z\,dz$, where γ is the intersection curve of the cylinder $x^2+y^2=a^2$ with the plane $z-y=a$, oriented counterclockwise.

II) $\displaystyle\int_\gamma \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=(2xy+z^2, x^2, 2xz)$, where γ is the intersection of the plane $x=y$ with the sphere $x^2+y^2+z^2=a^2$, positively oriented.

III) $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = (y, z, x)$, where γ is the curve intersection of $x^2 + y^2 = 2x$ and $x = z$.

Solution: i) $-2\pi a^2$; *ii)* 0; *iii)* 0.

Problem 4.6 Find the value of b that minimizes the work done in moving a particle by the force field $\mathbf{F}(x, y) = (3y^2 + 2, 16x)$, from $(-1, 0)$ to $(1, 0)$, along the semiellipse $b^2x^2 + y^2 = b^2$, $y \geq 0$.

Solution: The work done is $W(b) = 4b^2 - 8\pi b + 4$ and the minimum work is $4 - 4\pi^2$, obtained for $b = \pi$.

Problem 4.7 Consider the force field $\mathbf{F}(x, y) = (cxy, x^6y^2)$, $a, b, c > 0$. Find the parameter a in terms of c such that the work done in moving a particle along the parabola $y = ax^b$ from $x = 0$ to $x = 1$ will not depend on b.

Solution: the work is $\frac{3ac+a^3b}{3(b+2)}$, hence, $a = 0$ or $a = \sqrt{3c/2}$.

Problem 4.8 Evaluate the work done in moving a particle under a force field (given in polar coordinates) $\mathbf{F}(r, \theta) = (-4\sin\theta, 4\sin\theta)$, along the path $r = \mathrm{e}^{-\theta}$ from (1,0) to the origin.

Solution: 8/5.

Problem 4.9 Let $\mathbf{F}(x, y, z) = (\sin y + z, x\cos y + \mathrm{e}^z, x + y\mathrm{e}^z)$.

I) Prove that the integral over any piecewise C^1 simple closed curve is equal to 0.

II) Obtain a potential of F, that is, find ϕ such that $\mathbf{F} = \nabla\phi$.

Solution: ii) $\phi(x, y, z) = x(\sin y + z) + y\mathrm{e}^z$.

Problem 4.10 Evaluate $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = (2xz\mathrm{e}^{x^2+y^2}, 2yz\mathrm{e}^{x^2+y^2}, \mathrm{e}^{x^2+y^2})$ and γ the path on $\mathbb{R}^3$ given by $\mathbf{r}(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$.
Hint: Prove that $\mathbf{F}$ is a gradient field.

Solution: e^2.

Problem 4.11 Given the curve on $\mathbb{R}^3$, $\gamma(t) = \left(\mathrm{e}^{t^2} + t(1-\mathrm{e}) - 1, \sin^5(\pi t), \cos(t^2 - t)\right)$, $t \in [0, 1]$, and the vector field

$$\mathbf{F}(x, y, z) = (y + z + x^4 \sin x^5,\ x + z + \operatorname{arc\,tg} y,\ x + y + \sin^2 z).$$

I) Find $\int_\gamma \mathbf{F}$.

II) Does it exist f such that $\nabla f = \mathbf{F}$? If this is the case, find f.

Solution: i) 0; *ii)* $f(x, y, z) = xy + xz + yz - \frac{1}{5}\cos x^5 + y \operatorname{arc\,tg} y - \frac{1}{2}\log(1 + y^2) + \frac{z}{2} - \frac{1}{4}\sin 2z$.

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Problem 4.12 Given the curve on $I\!R^3$, $\Gamma = \{\, x^2 + y^2 = 1,\ z = y^2 - x^2 \,\}$, positively oriented, and the vector field $\mathbf{F}(x, y, z) = (y^3, \mathrm{e}^y, z)$.

I) Find $\displaystyle\int_\Gamma \mathbf{F}$.

II) Does it exist f such that $\nabla f = \mathbf{F}$?

Solution: i) $-3\pi/4$; *ii)* No.

Problem 4.13 Determine a and b such that the vector field

$$\mathbf{w}(x, y) = \mathrm{e}^{2x+3y}\Big(a \sin x + a \cos y + \cos x,\ b \sin x + b \cos y - \sin y\Big)$$

is irrotational (that is, its curl is 0) and find its potential.

Solution: $a = 2,\ b = 3;\ \varphi(x, y) = \mathrm{e}^{2x+3y}(\sin x + \cos y) + C$.

Problem 4.14 Consider the vector field

$$\mathbf{F}(x, y) = \Big(\frac{\log x + \log y}{x}\,,\ \frac{\log x + \log y}{y}\Big),$$

defined on the domain $D = \{(x, y) :\ x > 0\,,\ y > 0\}$.

I) Evaluate $\displaystyle\int_\gamma \mathbf{F}$, where γ is the arc of the hyperbola $xy = a$ $(a > 0)$, such that $x_1 \le x \le x_2$.

II) Let A be any point of the hyperbola $xy = a$ $(a > 0)$, B any point of the hyperbola $xy = b$ $(b > a)$, and γ any C^1 path, contained on D joining A to B, prove that

$$\int_\gamma \mathbf{F} = \frac{1}{2}\log\frac{b}{a}\log(ab)\,.$$

Hint: $\mathbf{F}$ is conservative.

Solution: i) 0.

Problem 4.15 Evaluate $\displaystyle\int_\gamma (5 - xy - y^2)dx - (2xy - x^2)dy$, where γ is the square of vertices (0,0), (1,0), (1,1) and (0,1), compute it directly and also applying Green's Theorem.

Solution: 3/2.

Problem 4.16 Let f be a C^1 function on $I\!R$. Let

$$P(x, y) = \mathrm{e}^{x^2} - \frac{y}{3 + \mathrm{e}^{xy}},\quad Q(x, y) = f(y),$$

and γ the boundary of the square $[0, 1] \times [0, 1]$ oriented in the positive direction. Evaluate $\displaystyle\int_\gamma P dx + Q dy$.

Solution: $(1 - \log(\mathrm{e} + 3) + \log 4)/3$.

Problem 4.17 Evaluate $\int_\Gamma xy\,dx + \sin^2(e^{\cos y})\,dy$, where Γ is the curve $y = e^{-x^2}$, for $x \in (-\infty, \infty)$.
Hint: Apply Green's Formula to the same integral over the curve Γ_R, formed by the line segment $(-R, R)$, the function $y = e^{-x^2}$ on the same interval and the vertical line segments joining both of them, positively oriented ; after, take the limit when $R \to \infty$.

Solution: 0.

Problem 4.18 Let the functions $P(x, y) = y/(x^2 + y^2)$ and $Q(x, y) = -x/(x^2 + y^2)$. Let C be a piecewise C^1 closed curve, defined outside the origin, such that is the boundary of a region D.

I) Prove that $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ for $(x, y) \neq (0, 0)$.

II) If $(0, 0) \in D$, prove that $\int_C P\,dx + Q\,dy = \pm 2\pi$.

III) If $(0, 0) \notin D$, compute $\int_C P\,dx + Q\,dy$.

Solution: iii) 0.

Problem 4.19 Evaluate $\displaystyle\int_\gamma \frac{-y\,dx + (x-1)\,dy}{(x-1)^2 + y^2}$, where γ is a piecewise C^1 simple closed curve, containing (1,0) in its interior, oriented in the positive direction.

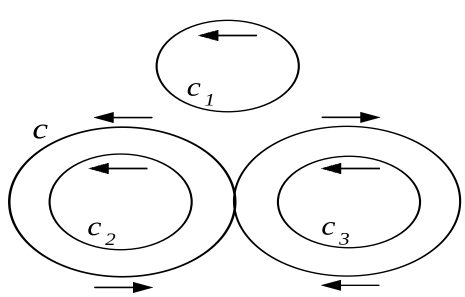
Solution: 2π.

Problem 4.20 Let $P, Q \in C^1(\mathbb{R}^2)$ be two scalar fields such that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ everywhere on the plane but at three points. Let C_1, C_2 and C_3 be three disjoint circles surrounding them and $I_k = \int_{C_k} P\,dx + Q\,dy$. If $I_1 = 12$, $I_2 = 10$ and $I_3 = 15$,

I) compute $\int_C P\,dx + Q\,dy$, where C is the curve of the figure, surrounding C_2 and C_3;

II) sketch γ, such that $\int_\gamma P\,dx + Q\,dy = 1$;

III) if $I_1 = 12$, $I_2 = 9$ and $I_3 = 15$, prove that is impossible to find such a curve γ.



Hint: *iii)* the integral is proportional to 3.
Solution: i) -5; *ii)* γ is any curve surrounding once C_2 in the positive direction, once C_3 in the positive direction and twice C_1 in the negative direction.

Problem 4.21

I) Let A be the area of a region D, bounded by C, a piecewise C^1 simple closed curve. Prove that

$$A = \frac{1}{2}\int_C -y\,dx + x\,dy = \int_C x\,dy = -\int_C y\,dx\,,$$

and prove also that in polar coordinates it takes the form

$$A = \frac{1}{2}\int_C r^2(\theta)\,d\theta.$$

II) Evaluate the area of the interior of the loop of the curve parametrized as $\mathbf{s}(t) = (t^2-1, t^3-t)$.

III) Evaluate the area of the cardioid, given in polar coordinates as $r(\theta) = a(1-\cos\theta)$, $(0 \le \theta \le 2\pi)$.

Solution: ii) 8/15; *iii)* $3\pi a^2/2$.

Problem 4.22

I) Evaluate $\int_D (x+2y)dxdy$, where D is the cycloid's arc $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$.

II) Evaluate $\int_D xy^2 dxdy$, where D is the region bounded by the astroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \pi/2$ and the coordinate axes.

III) Evaluate $\int_D y^2 dxdy$, where D is the region bounded by the curve $x = a(t - \sin^2 t)$, $y = a\sin^2 t$, $0 \le t \le \pi$, and the line joining its endpoints.

Solution: i) $\pi(3\pi+5)$; *ii)* 8/2145; *iii)* $5\pi a^4/48$.

Problem 4.23 Let $a, b > 0$.

I) Prove (by integrating) that:

$$\int_0^{\pi/2} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{\pi}{2ab}\,.$$

II) Using part *i)* and taking derivatives inside the integral, prove that

$$\int_0^{\pi/2} \frac{\sin^2 t}{(a^2\cos^2 t + b^2\sin^2 t)^2}\,dt = \frac{\pi}{4ab^3}\,.$$

III) Using part *ii)*, prove that

$$\int_\gamma \frac{y^3\,dx - xy^2\,dy}{(x^2+y^2)^2} = \pi\,,$$

where γ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ clockwise oriented.

Problem 4.24 Let $r = \|\mathbf{x}\|$, defined for all $\mathbf{x} \in I\!R^n$, .

I) Find ∇f for $f(\mathbf{x}) = r^\alpha$.

II) Do the same for $f(\mathbf{x}) = g(r)$, where g is a differentiable function of one variable.

III) Find div $\mathbf{F}$ for $\mathbf{F}(\mathbf{x}) = r^\alpha \mathbf{x}$.

IV) Find $\Delta f = \text{div}\,(\nabla f)$ for $f(\mathbf{x}) = r^\alpha$.

V) Find a potential for the force field $\mathbf{F}(\mathbf{x}) = g(r)\mathbf{x}$, where g is a continuous function of one variable.

Solution: i) $\alpha r^{\alpha-2}\mathbf{x}$; *ii)* $g'(r)\mathbf{x}/r$; *iii)* $(\alpha+n)r^\alpha$; *iv)* $\alpha(\alpha+n-2)r^{\alpha-2}$; *v)* $\varphi(r) = \int_0^r sg(s)ds$.

Problem 4.25 Let D be a region defined in $I\!R^2$ bounded by the regular closed curve C, and let $u, v \in C^2(\overline{D})$. If $\mathbf{n}$ denotes the unit normal vector exterior to the curve, use Divergence's Theorem to prove the following identities:

$$i) \qquad \int_C \frac{\partial u}{\partial \mathbf{n}}\, ds = \int_D \Delta u\, dxdy$$

$$ii) \qquad \int_C v\frac{\partial u}{\partial \mathbf{n}}\, ds = \int_D (v\Delta u + \nabla u \cdot \nabla v)\, dxdy$$

$$iii) \qquad \int_C \left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\right) ds = \int_D (v\Delta u - u\Delta v)\, dxdy\,.$$

Problem 4.26 Let u and v be two C^1 class scalar fields defined on an open containing the unit disk D. If $\mathbf{F}(x,y) = (v(x,y), u(x,y))$ and $\mathbf{G}(x,y) = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right)$. Evaluate

$$\int_D \mathbf{F} \cdot \mathbf{G}\, dxdy\,,$$

where u and v verifies that $u = 1$ and $v = y$ at the unit circumference.

Solution: $-\pi$.