5. SURFACE INTEGRALS

Parametrizations of important surfaces:

Sphere:	$x^2+y^2+z^2=R^2$	$\Rightarrow$	$\phi_1(\theta,\varphi)=(R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi)$.
Cylinder:	$x^2+y^2=R^2$	$\Rightarrow$	$\phi_2(\theta,z)=(R\cos\theta, R\sin\theta, z)$.
Cone:	$\sqrt{x^2+y^2}=z$	$\Rightarrow$	$\phi_3(r,\theta)=(r\cos\theta, r\sin\theta, r)$.
Paraboloid:	$x^2+y^2=z$	$\Rightarrow$	$\phi_4(r,\theta)=(r\cos\theta, r\sin\theta, r^2)$.
Helicoid:		$\Rightarrow$	$\phi_5(r,\theta)=(r\cos\theta, r\sin\theta, \theta)$

Problem 5.1 Sketch the graphs of the previous surfaces.

Problem 5.2 Evaluate the area of the following surfaces:

I) sphere of radius R;

II) circular cone parametrized by $\mathbf{r}(u,v)=(u\cos v, u\sin v, u)$, where $0\le u\le a$ and $0\le v\le 2\pi$.

III) portion of the paraboloid $z=x^2+y^2$ inside the cylinder $x^2+y^2=a^2$;

IV) portion of the cylinder $x^2+z^2=16$ bounded by the cylinder $x^2+y^2=16$.

Solution: i) $4\pi R^2$; *ii)* $\pi a^2\sqrt{2}$; *iii)* $\pi((1+4a^2)^{3/2}-1)/6$; *iv)* 128.

Problem 5.3

I) Deduce the area's formula of the revolution surface obtained by rotating the graph $y=f(x)$, $0<a\le x\le b$, about the vertical axis:

$$A=2\pi\int_a^b x\sqrt{1+(f'(x))^2}\,dx\,,$$

using the parametrization $\mathbf{s}(r,\theta)=(r\cos\theta, r\sin\theta, f(r))$, where $a\le r\le b$ and $0\le\theta\le 2\pi$.

II) Find the surface area of the torus obtained by rotating about the vertical axis the graph $(x-R)^2+y^2=c^2$, $0<c<R$.

III) Deduce the corresponding parametrization to obtain the analogous formula in the case of rotating the graph $y=f(x)$, $a\le x\le b$, about the horizontal axis.

Solution: ii) $4\pi^2 Rc$; *iii)* $\mathbf{s}(x,\theta)=(x, f(x)\cos\theta, f(x)\sin\theta)$).

Problem 5.4 Let $W=\{1\le z\le (x^2+y^2)^{-1/2}\}$ be a region of $\mathbb{R}^3$. Prove that W has finite volume but its boundary has infinite area.

Solution: $V=\pi$.

Problem 5.5 Find the moment of inertia with respect to a diameter of an homogeneous spherical shell of mass m and radius a.

Solution: $2ma^2/3$.

Problem 5.6 Evaluate $\int_S \mathbf{F}\cdot\mathbf{n}\,dS$ ($\mathbf{n}$ the outward normal vector) in the following cases:

I) $\mathbf{F}(x,y,z)=(x^2,y^2,z^2)$ and S the boundary of the cube $0\le x,\ y,\ z\le 1$.

II) $\mathbf{F}(x,y,z)=(xy,-x^2,x+z)$ and S the portion of the plane $2x+2y+z=6$ on the first octant, with $\mathbf{n}$ the normal vector with positive third component.

III) $\mathbf{F}(x,y,z)=(xz^2,x^2y-z^2,2xy+y^2z)$ and S the upper hemisphere $z=\sqrt{a^2-x^2-y^2}$.

IV) $\mathbf{F}(x,y,z)=(2x^2+\cos yz, 3y^2z^2+\cos(x^2+z^2), e^{y^2}-2yz^3)$ and S the surface of the solid generated by intersecting the cone $z\ge\sqrt{x^2+y^2}$ with the ball $x^2+y^2+z^2\le 1$.

Solution: i) 3; *ii)* 27/4; *iii)* $2a^5\pi/5$; *iv)* 0.

Problem 5.7
Consider the surface

$$S=\{\,(x,y,z)\in I\!R^3\ :\ x^2+y^2+z^2=1,\ y\ge 0\,\}$$

(oriented with the normal vector exterior to the unit sphere) and the function

$$\mathbf{F}(x,y,z)=(x+z,y+z,2z).$$

I) Evaluate $\int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$.

II) Evaluate $\int_S \mathbf{F}\cdot\mathbf{n}\,dS$.

Solution: i) π; *ii)* $8\pi/3$.

Problem 5.8 Using Stokes's Theorem compute the integral $\displaystyle\int_S \operatorname{curl}\mathbf{F}$ in the following cases, where S is oriented with outward normal vector:

I) $\mathbf{F}(x,y,z)=(x^2y^2,yz,xy)$ and S the paraboloid $z=a^2-x^2-y^2,\ z\ge 0$.

II) $\mathbf{F}(x,y,z)=((1-z)y,ze^x,x\sin z)$ and S the upper hemisphere of radius a.

III) $\mathbf{F}(x,y,z)=(x^3+z^3,e^{x+y+z},x^3+y^3)$ and $S=\{\,x^2+y^2+z^2=1,\ y\ge 0\,\}$.

Solution: i) 0; *ii)* $-\pi a^2$; *iii)* 0.

Problem 5.9 Consider the vector field

$$\mathbf{F}(x,y,z)=\Big(y\,,\ x^2\,,\ (x^2+y^4)^{3/2}\,\sin(e^{\sqrt{xyz}})\Big)\,.$$

Evaluate $\displaystyle\int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ denotes the inward normal vector to the semiellipsoid

$$S=\{(x,y,z):\ 4\,x^2+9\,y^2+36\,z^2=36\,,\ z\ge 0\}\,.$$

Solution: 6π.

Problem 5.10 Let $\mathbf{F}(x,y,z) = (2y, 3z, x)$ and T be the triangle with vertices $A(0,0,0)$, $B(0,2,0)$ and $C(1,1,1)$.

I) Find an orientation of the triangle's surface and the one induced at the boundary.

II) Evaluate the line integral of the vector field $\mathbf{F}$ over the boundary of T.

Solution: *i)* $\mathbf{n} = (1,0,-1)$; the orientation of the boundary is $A \to B \to C \to A$; *ii)* -1.

Problem 5.11 Redo parts *i)*, *iii)* and *iv)* of problem 5.6, using the vector calculus theorems.

Problem 5.12 Consider the function

$$\mathbf{F}(x,y,z) = (y\sin(x^2+y^2), -x\sin(x^2+y^2), z(3-2y))$$

and the region

$$W = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2+z^2 \leq 1, z \geq 0\}.$$

Evaluate $\int_{\partial W} \mathbf{F}$, where ∂W denotes the boundary of W.

Solution: 2π.

Problem 5.13 Verify Stokes's Theorem for:

I) $\mathbf{F}(x,y,z) = (y^2, xy, xz)$, on the paraboloid $z = a^2 - x^2 - y^2,\ z \geq 0$.

II) $\mathbf{F}(x,y,z) = (-y^3, x^3, z^3)$ on $S = \{\, z = y,\ y \geq 0,\ x^2+y^2 \leq 1 \,\}$.

Solution: *i)* 0; *ii)* $3\pi/4$.

Problem 5.14 Evaluate the integral $\int_S \mathbf{F}$, where

I) $\mathbf{F}(x,y,z) = (18z, -12, 3y)$, and S is the region of the plane $2x+3y+6z = 12$ on the first octant.

II) $\mathbf{F}(x,y,z) = (x^3, x^2y, x^2z)$, and S is the closed surface formed by the cylinder $x^2+y^2 = a^2,\ 0 \leq z \leq b$, and its upper and lower bases.

III) $\mathbf{F}(x,y,z) = (4xz, -y^2, yz)$, and S is the boundary surface of the cube $0 \leq x,\ y,\ z \leq 1$.

IV) $\mathbf{F}(x,y,z) = (x,y,z)$, and S is a simple closed surface.

Solution: *i)* 24; *ii)* $5\pi a^4 b/4$; *iii)* $3/2$; *iv)* $3|\Omega|$, where $S = \partial\Omega$.

Problem 5.15 Compute the flux of the vector field $\mathbf{F}(x,y,z) = (y^2, yz, xz)$ over the surface of the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$, $x+y+z = 1$, oriented with outward normal.

Solution: $1/12$.

Problem 5.16 Suppose a temperature function on each point of the space is proportional to the square of the vertical axis distance. Consider the region $V = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 2z,\ z \leq 2 \}$.

I) Compute the volume of V.

II) Compute the average temperature on V.

III) Compute the gradient flux of the temperature out of ∂V.

Solution: i) 4π; *ii)* $4\alpha/3$ (if α is the constant of proportionality); *iii)* $16\alpha\pi$.

Problem 5.17 Let D be a region on the plane with boundary $C = \partial D$. Suppose that the inertia moments of D with respect to the coordinate axes are $I_x = a$ and $I_y = b$ respectively. If the density is 1, $\mathbf{n}$ is the outward unit normal vector to C, and $r = \|\mathbf{x}\| = \sqrt{x^2 + y^2}$, compute the integral

$$\int_C \nabla r^4 \cdot \mathbf{n}\, ds.$$

Solution: $16(a + b)$.

Problem 5.18 Let S be the unit sphere on $\mathbb{R}^3$, and let φ be a nonzero function verifying $\|\nabla\varphi\|^2 = 4\varphi$ and $\operatorname{div}(\varphi\nabla\varphi) = 10\varphi$. Evaluate

$$\int_S \frac{\partial\varphi}{\partial\mathbf{n}}\, dS,$$

where $\mathbf{n}$ is the outward unit normal vector to S.
Hint: From the two equations involving φ, compute $\Delta\varphi$.

Solution: 8π.

Problem 5.19 Let S be the sphere of radius a oriented with its outward normal vector, and let the vector field $\mathbf{F}(x, y, z) = (\sin yz + e^z, x\cos z + \log(1 + x^2 + z^2), e^{x^2+y^2+z^2})$. Evaluate $\int_S \mathbf{F}$.

Solution: 0.

Problem 5.20 Let $S = S_1 \cup S_2$, where S_1 and S_2 are the surfaces

$$S_1 = \{ x^2 + y^2 = 1,\ 0 \leq z \leq 1 \}, \qquad S_2 = \{ x^2 + y^2 + (z - 1)^2 = 1,\ z \geq 1 \},$$

and let the vector field $\mathbf{F}(x, y, z) = (zx + z^2y + x, z^3yx + y, z^4x^2)$.

I) Evaluate $\int_S \operatorname{curl} \mathbf{F}$ using Stokes's Theorem.

II) Evaluate the same integral using Divergence's Theorem.

Solution: 0.

Problem 5.21 Let $h : I\!R^2 \to I\!R$ be a C^1 function. Find $\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ is the inward unit normal vector to $\partial\Omega$, and

$$\mathbf{F}(x, y, z) = \left(\mathrm{e}^{y^2+z^2} + \int_0^x \frac{\mathrm{e}^{t^2+y^2}}{\sqrt{t^2+y^2}}\, dt,\ \sin(x^2 + \mathrm{e}^z),\ h(x, y) \right),$$

$$\Omega = \{ (x, y, z) \in I\!R^3 \,:\, x^2 + y^2 \le 1,\ 0 \le z \le \sqrt{x^2+y^2},\ x \ge 0,\ y \ge 0 \}.$$

Solution: $\pi(1 - \mathrm{e})/4$.

Problem 5.22 Consider the vector field

$$\mathbf{F}(x, y, z) = \left(y\,\mathrm{e}^z\,,\ \int_0^x \mathrm{e}^{-t^2+\cos z} dt\,,\ z(x^2+y^2) \right).$$

Evaluate $\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the outward normal vector to the boundary of the region

$$\Omega = \{(x, y, z) :\ x^2 + y^2 + z^2 < a^2\,,\ x^2 + y^2 < z^2\}\,.$$

Solution: $(8 - 5\sqrt{2})\pi a^5/15$.

Problem 5.23 A vector field on $I\!R^3$ has the form $\mathbf{F}(x, y, z) = (P_1(x, y) + P_2(x, z), x + Q(y, z), R(x, y, z))$, with $P_1, P_2, Q, R \in \mathcal{C}^2(I\!R^3)$. If Γ_h is the boundary of the cylinder's section $x^2 + y^2 = 1$ at height h, prove that $\int_{\Gamma_h} \mathbf{F} \cdot d\mathbf{r}$ is independent of h.