Universidad Carlos III de Madrid

Escuela Politécnica Superior

DEPARTAMENTO DE MATEMÁTICAS
First Course. Telecommunication Engineering

Calculus II. Final exam. May 2010 **Time length 3 h 30 min.**

Problem 1. (2.5 p.) Analize the convergence of the following improper integrals depending on the given parameters:

a) $\displaystyle\int_1^\infty \frac{dx}{x^\alpha \sqrt[3]{1+x^3}},$ for $\alpha \in \mathbb{R}$,

b) $\displaystyle\int_0^1 x^p(1-x)^q\,dx,$ for $p, q \in \mathbb{R}$.

Problem 2. (2.5 p.) Let $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4,\ x, y \ge 0\}$. Compute the following integral

$$\iint_D \frac{\log(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\,dxdy.$$

Problem 3. (2.5 p.) Given the force field in $\mathbb{R}^2$,

$$\mathbf{F}(x,y) = \left(\frac{\pi}{2}\cos(\pi x/2)e^{\sin(\pi x/2)} + 2xy^2, 2yx^2\right).$$

(a) Determine if the field is conservative. In case it is conservative, find its potential function.

(b) Compute the work done by $\mathbf{F}$ in moving a particle along the curve $\{y = 1+x, x \in [-1,0]\} \cup \{y = 1-x, x \in [0,1]\}$, from $(-1,0)$ to $(1,0)$.

Problem 4. (2.5 p.) Verify the Gauss's Divergence Theorem to compute the outer flux through the unit sphere of the following vector field

$$\mathbf{F}(x,y,z) = (y, -x, z).$$

---

1. a) Comparing with the function $g(x) = \dfrac{1}{x^{\alpha+1}}$, that converges for $\alpha > 0$:

$$\lim_{x\to\infty} \frac{\dfrac{1}{x^\alpha \sqrt[3]{1+x^3}}}{\dfrac{1}{x^{\alpha+1}}} = \lim_{x\to\infty} \frac{x}{\sqrt[3]{1+x^3}} = \lim_{x\to\infty} \frac{1}{\sqrt[3]{1/x^3+1}} = 1,$$

therefore, by the limit comparison test, the integral $\boxed{\text{converges for } \alpha > 0}$ and diverges for $\alpha \leq 0$.

b) Breaking the integral as $\displaystyle\int_0^1 x^p(1-x)^q\,dx = \int_0^{1/2} x^p(1-x)^q\,dx + \int_{1/2}^1 x^p(1-x)^q\,dx$.
The first one is improper at $x = 0$. We compare it with x^p that converges for $p > -1$:

$$\lim_{x\to 0} \frac{x^p(1-x)^q}{x^p} = 1,$$

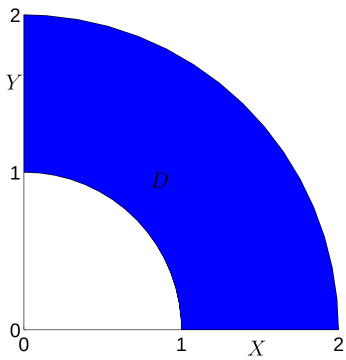
thus, both have the same behaviour, so it converges for $p > -1$.
The second one is improper at $x = 1$. With the change of variables $t = 1 - x$,
$\displaystyle\int_{1/2}^1 x^p(1-x)^q\,dx = \int_0^{1/2} t^q(1-t)^p\,dt$, the integral becomes as the previous one changing the roles of p and q, so this integral converges for $q > -1$.
Therefore, the integral converges for $\boxed{p, q > -1.}$

2.

$$\iint_D \frac{\log(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\,dxdy \underset{polar}{=} \int_0^{\pi/2} d\theta \int_1^2 \frac{r\log r}{r}\,dr =$$

$$= \frac{\pi}{2}\int_1^2 \log r\,dr = \frac{\pi}{2}\Big[r\log r - r\Big]_1^2 = \boxed{\frac{\pi}{2}\,[2\log 2 - 1].}$$

To compute $\int_1^2 \log r\,dr$ we do integration by parts with $u = \log r$ and $dv = dr$.



---

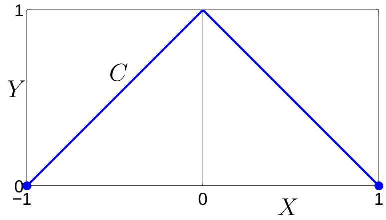
3. a) If we denote $\mathbf{F} = (P, Q)$, since $\dfrac{\partial P}{\partial y} = 4xy = \dfrac{\partial Q}{\partial x}$, the force field is conservative.

The potential f is obtained from $\mathbf{F} = \boldsymbol{\nabla} f$. The second equation gives us $\dfrac{\partial f}{\partial y} = 2yx^2 \Rightarrow$ $f = y^2x^2 + g(x)$, where $g(x)$ is a function depending only on the variable x. From the first equation:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xy^2 + g'(x) = \frac{\pi}{2}\cos(\pi x/2)e^{\sin(\pi x/2)} + 2xy^2 \Rightarrow \\
g'(x) &= \frac{\pi}{2}\cos(\pi x/2)e^{\sin(\pi x/2)} \Rightarrow \\
g(x) &= \int \frac{\pi}{2}\cos(\pi x/2)e^{\sin(\pi x/2)}\,dx = e^{\sin(\pi x/2)} + C.
\end{aligned}$$

Therefore, the potential is $\boxed{f(x,y) = e^{\sin(\pi x/2)} + y^2x^2 + C.}$

b) As the force field is conservative, the work done is



$$W = \int_l \mathbf{F}\cdot d\mathbf{r} = f(1,0) - f(-1,0) =$$
$$= e^{\sin(\pi/2)} - e^{\sin(-\pi/2)} = \boxed{e - e^{-1}.}$$

We can obtain also the work without using the potential, using the fact that for two any trajectories with same endpoints the work is the same. Thus, choosing the following, $\mathbf{r} = (x, 0)$, $x \in [-1, 1]$, the work done is:

$$W = \int_{-1}^{1} \Big(\frac{\pi}{2}\cos(\pi x/2) e^{\sin(\pi x/2)}, 0\Big)\cdot(1,0)\, dx = e^{\sin(\pi x/2)}\Big|_{-1}^{1} = e - e^{-1}.$$

4. We have to verify the Gauss's Divergence Theorem

$$\iiint\limits_{\Omega} \operatorname{div}\mathbf{F}\, dv = \iint\limits_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S},$$

so we have to calculate both integrals.

$$\iiint\limits_{\Omega} \operatorname{div}\mathbf{F}\, dv = \iiint\limits_{\Omega} 1\, dV \underset{\text{spherical}}{=} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^1 r^2\, dr = 2\pi\Big[-\cos\phi\Big]_0^{\pi} \frac{r^3}{3}\Big|_0^1 = \boxed{\frac{4\pi}{3}.}$$

To compute this integral, we need to parametrize the surface $\partial\Omega$ of the unit sphere:
$\mathbf{\Phi}(\phi, \theta) = (\cos\theta\, \sin\phi, \sin\theta\, \sin\phi, \cos\phi)$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$.
$\mathbf{\Phi}_\phi \times \mathbf{\Phi}_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\, \sin\theta, \sin\phi\cos\phi)$. Therefore, we have that

$$\begin{aligned}
\iint\limits_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi} d\theta \int_0^{\pi} \mathbf{F}(\mathbf{\Phi}(\phi,\theta))\cdot \mathbf{\Phi}_\phi \times \mathbf{\Phi}_\theta\, d\phi = \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} (\sin\theta\, \sin\phi, -\cos\theta\, \sin\phi, \cos\phi)\cdot(\sin^2\phi\cos\theta, \sin^2\phi\, \sin\theta, \sin\phi\cos\phi)\, d\phi = \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} \cos^2\phi\, \sin\phi\, d\phi = 2\pi\Big[-\frac{\cos^3\phi}{3}\Big]_0^3 = \boxed{\frac{4\pi}{3}.}
\end{aligned}$$

Notice that we have chosen this orientation, that is the one corresponding with the outwards normal. This can be seen easily choosing the point $(1, 0, 0)$, that is, for the values of $\theta = 0$ and $\phi = \pi/2$, so we get $\mathbf{\Phi}_\phi \times \mathbf{\Phi}_\theta = (1, 0, 0)$, that is the correct one.