Universidad Carlos III de Madrid

Escuela Politécnica Superior

Departamento de Matemáticas

First Course. Telecommunication Engineering

Calculus II. Final exam. June 2010 **Time length 3 h 30 min.**

Problema 1. (2.5 p.) Solve the following initial value problem

$$y'' - 8y' + 16y = t^2 e^{4t},$$
$$y(0) = 3, \; y'(0) = 12.$$

Problema 2. (2.5 p.) Let $K = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1, \; z \geq 0\right\}$, find the integral

$$\iiint_K z \, dx \, dy \, dz.$$

Problema 3. (2.5 p.) Verify Green's Theorem to compute the line integral of the function $\mathbf{F}(x, y) = (-y + 1, x^2 + y^2)$, along the curve $\{x^2 + y^2 = 1, y > 0\} \cup \{y = 0, x \in [-1, 1]\}$ positively oriented, that is, counterclockwise.

Problema 4. (2.5 p.) Let C be the intersection curve of the plane $y + z = 2$ with the cylinder $x^2 + y^2 = 1$, oriented positively when we project it onto the plane $z = 0$. Compute the line integral

$$\int_C -y^2 dx + x dy + z^2 dz.$$

---

1. If we do the Transform of Laplace to the equation, we get

$$s^2F(s) - sy(0) - y'(0) - 8(sF(s) - y(0)) + 16F(s) = L[t^2e^{4t}](s) = \frac{2!}{(s-4)^3} =$$

$$= F(s)(s^2 - 8s + 16) - 3s - 12 + 24 = \frac{2}{(s-4)^3} \to F(s) = \frac{3}{s-4} + \frac{2}{(s-4)^5} \to$$

$$\boxed{y(t) = e^{4t}\Big(3 + \frac{t^4}{12}\Big).}$$

---

2. With the following changes of variables:

First C. V.: $\begin{cases} x = u, \\ y = 2v, \\ z = 3w \end{cases}$ $J = 6$. On the region $K' = \{u^2 + v^2 + z^2 \leq 1,\ w \geq 0\}$.

Second C. V., to spherical coordinates: $\begin{cases} u = r\cos\theta\sin\phi, \\ v = r\sin\theta\sin\phi, \\ w = r\cos\phi, \end{cases}$ $|J| = r^2\sin\phi$,

$r \in [0,1],\ \theta \in [0, 2\pi],\ \phi \in [0, \pi/2]$. The integral becomes

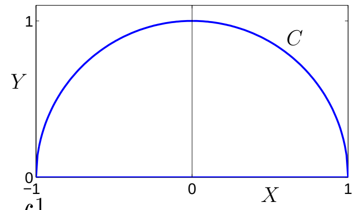
$$\iiint_K z\,dx\,dy\,dz = 18\iiint_{K'} w\,du\,dv\,dw = 18\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\cos\phi\,d\phi \int_0^1 r^3 dr =$$

$$18 \cdot 2\pi \frac{\sin^2\phi}{2}\Big|_0^{\pi/2} \frac{r^4}{4}\Big|_0^1 = \boxed{\frac{9\pi}{2}.}$$

---

3. Green's Theorem says that

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$



Therefore, we must compute both integrals:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_D (2x+1)\ dxdy \underset{C.V.polar}{=} \int_0^{\pi} d\theta \int_0^1 (2r\cos\theta + 1)r\,dr =$$

$$= 2\int_0^{\pi} \cos\theta\,d\theta \int_0^1 r^2\,dr + \int_0^{\pi} d\theta \int_0^1 r\,dr = -2\,[\sin\theta]_0^{\pi} \cdot \left[\frac{r^3}{3}\right]_0^1 + \pi\left[\frac{r^2}{2}\right]_0^1 = \boxed{\frac{\pi}{2}.}$$

For the other integral, we break the parametrization of the curve in two parts:
$\begin{cases} C_1 \to r_1 = (x, 0),\ x \in [-1, 1] \\ C_2 \to r_2 = (\cos t, \sin t),\ t \in [0, \pi] \end{cases}$, so we have that

$$\int_C P\,dx + Q\,dy = \int_{-1}^1 (1, x^2)\cdot(1,0)\,dx + \int_0^{\pi} (-\sin t + 1, 1)\cdot(-\sin t, \cos t)\,dt =$$

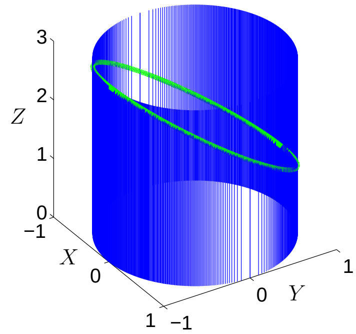
$$= \int_{-1}^1 dx + \int_0^{\pi} (\sin^2 t - \sin t + \cos t)\ dt = \int_{-1}^1 dx + \int_0^{\pi}\left(\frac{1-\cos 2t}{2} - \sin t + \cos t\right) dt =$$

$$= [x]_{-1}^1 + \left[\cos t + \sin t + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} = 2 - 2 + \frac{\pi}{2} = \boxed{\frac{\pi}{2}.}$$

---

4. Using Stokes's Theorem the integral becomes easier:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}, \text{ where } C = \partial S.$$



Since $\operatorname{curl}\mathbf{F} = \operatorname{curl}(-y^2, x, z^2) = (0,0,1+2y)$.
We parametrize the surface as
$\mathbf{\Phi} = (r\cos\theta, r\sin\theta, 2 - r\sin\theta),\ r\in[0,1],\ \theta\in[0,2\pi]$, thus,
$\mathbf{\Phi}_r \times \mathbf{\Phi}_\theta = (0,r,r)$, that has the proper orientation. Therefore, the integral reads

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi} d\theta \int_0^1 (0,0,1+2r\sin\theta)\cdot(0,r,r)\,dr =$$

$$= 2\int_0^{2\pi}\sin\theta\,d\theta\int_0^1 r^2\,dr + \int_0^{2\pi}d\theta\int_0^1 r\,dr = 0 + 2\pi\left[\frac{r^2}{2}\right]_0^1 = \boxed{\pi.}$$