Universidad Carlos III de Madrid

Escuela Politécnica Superior

DEPARTAMENTO DE MATEMÁTICAS
First Course. Telecommunication Engineering

Calculus II. Third Test, May 11th, 2010

GRADE

Surname .... Name ....
D.N.I. .... Group ....

Time length: 80 min.

1. Consider a thread whose shape is the circumference arc $\{x^2+y^2=1\}$, from $\left(\sqrt{2}/2, \sqrt{2}/2\right)$ to $(-1,0)$. Determine its mass if the mass density is $\rho(x,y)=3x^2y$.

[2 p.]

2. Consider the force field in $\mathbb{R}^2$

$$F(x,y)=\left(e^x-\frac{y^2}{3+xy}, 4y\right).$$

Let γ be the line segment $y=x$ from $(0,0)$ to $(1,1)$. Find the work done in moving a particle along γ.

[2 p.]

3. Compute, using a surface integral, the area of the surface given by

$$\{(x,y,z)\in\mathbb{R}^3 : z=10-\sqrt{x^2+y^2}, z\geq 0\}.$$

[2 p.]

4. Let $S=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=9, z\geq 0\}$ be the surface oriented with the normal vector with negative third component. Consider the vector field

$$F(x,y,z)=\big((1-z)y, ze^x, x\sin z\big).$$

Compute

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{s}.$$

[2 p.]

5. Let S be the surface bounding the cube $\{(x,y,z)\in\mathbb{R}^3 : 0\leq x,y,z\leq 2\}$. Consider the vector field $\mathbf{F}(x,y,z)=\left(\frac{x^2}{2}, z+yxe^{x^2}, x+y\right)$. Find

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds,$$

where the normal $\mathbf{n}$ is oriented outwards.

[2 p.]

ANSWERS:

1. Parametrizing the circumference arc γ as $\mathbf{r}(t) = (\cos t, \sin t),\ t \in \left[\frac{\pi}{4}, \pi\right]$, the mass is

$$\begin{aligned} M &= \int_\gamma \rho(x,y)dr = \int_{\pi/4}^{\pi} \rho(\mathbf{r}(t))\|\mathbf{r}'(t)\| = \int_{\pi/4}^{\pi} 3\cos^2 t \sin t \sqrt{(-\sin(t))^2 + \cos^2 t} dt = \\ &= \int_{\pi/4}^{\pi} 3\cos^2 t \sin t = -\cos^3 t\Big|_{\pi/4}^{\pi} = \boxed{1 + \frac{\sqrt{2}}{4}.} \end{aligned}$$

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2. If we parametrize the curve γ as $\mathbf{r}(t) = (t,t),\ t \in [0,1]$, the work done is

$$\begin{aligned} W &= \int_\gamma \mathbf{F}\, d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^1 \left(e^t - \frac{t^2}{3+t^2}, 4t\right) \cdot (1,1)dt = \int_0^1 \left(e^t - \frac{t^2}{3+t^2} + 4t\right) dt = \\ &= \int_0^1 \left(e^t - 1 + \frac{3}{3+t^2} + 4t\right) dt = \left[e^t - t + 2t^2 + \frac{3}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}}\right]_0^1 = \boxed{e + \sqrt{3}\arctan \frac{1}{\sqrt{3}}.} \end{aligned}$$

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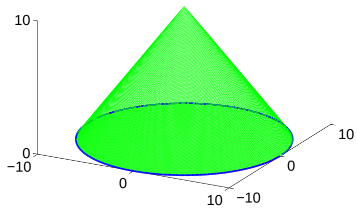
3. Parametrizing the surface S as
$\boldsymbol{\phi} = (r\cos\theta, r\,\mathrm{sen}\,\theta, 10 - r),\ r \in [0,10], \theta \in [0, 2\pi]$.
Therefore,
$\boldsymbol{\phi}_r \times \boldsymbol{\phi}_\theta = (\cos\theta, \mathrm{sen}\,\theta, -1) \times (-r\,\mathrm{sen}\,\theta, r\cos\theta, 10 - r) =$
$= (r\cos\theta, r\,\mathrm{sen}\,\theta, r)$ and $\|\boldsymbol{\phi}_r \times \boldsymbol{\phi}_\theta\| = \sqrt{2}r$. Then, the area is



$$A = \iint_S 1 ds = \int_0^{2\pi} \int_0^{10} \|\boldsymbol{\phi}_r \times \boldsymbol{\phi}_\theta\| dr d\theta = \int_0^{2\pi} d\theta \int_0^{10} \sqrt{2} r dr = \sqrt{2}\, 2\pi \frac{r^2}{2}\Big|_{r=0}^{10} = \boxed{100\pi\sqrt{2}.}$$

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4. If we use Stokes's Theorem we will just have to parametrize the boundary curve of the surface, ∂S: $\mathbf{r}(t) = (3\cos t, 3\sin t, 0),\ t \in [2\pi, 0]$, with the orientation induced by the one on S. Thus,

$$\begin{aligned} \iint_S \mathrm{curl}\,\mathbf{F} \cdot \mathbf{n}\, ds &\underset{Stokes}{=} \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{0} (3\sin t, 0, 0) \cdot (-3\sin t, 3\cos t, 0) dt \\ &= -9 \int_{2\pi}^{0} \sin^2 t dt = 9 \int_{2\pi}^{0} \frac{\cos 2t - 1}{2} dt = \boxed{9\pi.} \end{aligned}$$

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5. Let us denote as Ω the solid bounded by the surface. Then, by Gauss's Theorem we have that

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, ds &\underset{Gauss}{=} \iiint_\Omega \mathrm{div}\,\mathbf{F}\, dV = \iiint_\Omega (x + xe^{x^2})\, dV \\ &= \int_0^2 dz \int_0^2 dy \int_0^2 (x + xe^{x^2}) dx = 4\left[\frac{x^2}{2} + \frac{e^{x^2}}{2}\right]_0^2 = \boxed{6 + 2e^4.} \end{aligned}$$
