



Boolean Algebra and logic gates

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Outline

- Postulates and fundamental properties of Boolean Algebra
- Boolean functions and expressions
- Logic gates. Digital technologies. Implementation of logic functions
- Minimization of logic functions

Boolean Algebra

- Mathematical foundations of digital circuits
- Called Boolean Algebra after its inventor, George Boole
 - “*An Investigation of the Laws of Thought*” (1854)
- An algebra is defined by a set of elements with some operations. In our case:
 - $B = \{0, 1\}$
 - $\Phi = \{+, \cdot\}$

Postulates of Boolean Algebra

- Internal composition law
 - $\forall a, b \in B \Rightarrow a + b \in B, a \cdot b \in B$
- Neutral elements
 - $\forall a \in B \Rightarrow \exists$ neutral elements (0 y 1 respectively)
 $a + 0 = a$
 $a \cdot 1 = a$
- Commutative property
 - $\forall a, b \in B \Rightarrow a + b = b + a$
 $a \cdot b = b \cdot a$
- Distributive property
 - $\forall a, b, c \in B \Rightarrow a + b \cdot c = (a + b) \cdot (a + c)$
 $a \cdot (b + c) = a \cdot b + a \cdot c$



Postulates of Boolean Algebra

- Inverse or complementary element
 - $\forall a \in B \Rightarrow \exists \bar{a} \in B$

$$a + \bar{a} = 1$$

$$a \cdot \bar{a} = 0$$

Fundamental properties of Boolean Algebra

- Duality: Every valid law has its dual, which is obtained by replacing $0 \leftrightarrow 1$ and $+ \leftrightarrow \bullet$
- Idempotence

- $\forall a \in B \Rightarrow a + a = a$
 $a \bullet a = a$

- Proof:

$$a = a + 0 = a + a\bar{a} = (a + a)(a + \bar{a}) = (a + a) \bullet 1 = a + a$$

- Anihilation
 - $\forall a \in B \Rightarrow a + 1 = 1$
 $a \bullet 0 = 0$

Fundamental properties of Boolean Algebra

- The basic operations can be defined from the previous properties

a	b	$a+b$
0	0	0
0	1	1
1	0	1
1	1	1

a	b	$a \cdot b$
0	0	0
0	1	0
1	0	0
1	1	1

a	\bar{a}
0	1
1	0

- Truth table: provides the value of the function for all possible combinations of input values

Fundamental properties of Boolean Algebra

- Involution
 - $\forall a \in B \Rightarrow \bar{\bar{a}} = a$
- Absorption
 - $\forall a, b \in B \Rightarrow a + ab = a$
 $a(a+b) = a$
 - Proof:
$$a + ab = a \cdot 1 + ab = a(1 + b) = a \cdot 1 = a$$
- Associative property
 - $\forall a, b, c \in B \Rightarrow (a + b) + c = a + (b + c)$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Fundamental properties of Boolean Algebra

- De Morgan laws:

- $\forall a, b \in B \Rightarrow \overline{a + b} = \overline{a} \overline{b}$
$$\overline{a \cdot b} = \overline{a} + \overline{b}$$

- Proof:

$$(a + b) + \overline{a} \overline{b} = (a + b + \overline{a})(a + b + \overline{b}) = 1 \cdot 1$$

therefore $(a+b)$ is the inverse of $\overline{a} \overline{b}$



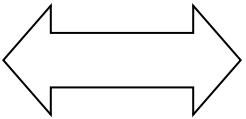
Boolean Functions and Expressions

- Definitions:
 - A logic or boolean variable is any element $x \in B = \{0, 1\}$
 - A literal is a variable or its inverse
 - Logic or Boolean Function:
$$f : B^n \rightarrow B$$
$$(x_1, x_2, \dots, x_n) \rightarrow y$$

Representation of logic functions

- Expression
- Truth table

$$f(a, b) = a + b$$

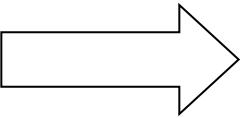


a	b	f(a,b)
0	0	0
0	1	1
1	0	1
1	1	1

Deriving the truth table from a expression

- Evaluate the expression for every combination of input values

$$f(a,b,c) = a + \bar{b}c$$

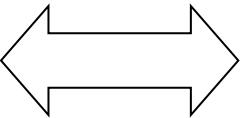


a	b	c	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Minterm function

- Expression: a product of all variables, either inverted or not
- Truth table: has a 1 for one combination and 0 elsewhere
- Example:

$$f(a,b,c) = \bar{a}\bar{b}\bar{c} = m_2$$



a	b	c	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

- Rule to obtain the expression:
 - 0 → inverted variable
 - 1 → non-inverted variable

Maxterm function

- Expression: a sum of all variables, either inverted or not
- Truth table: has a 0 for one combination and 1 elsewhere
- Example:

a	b	c	f
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$f(a,b,c) = (a + \bar{b} + c) = M_2$

↔

- Rule to obtain the expression:
 - 0 → inverted variable
 - 1 → non-inverted variable

ATTENTION: minterms use the opposite rule!

Shannon's Expansion Theorem

- Any boolean function can be decomposed in any of the following forms

$$f(x_1, x_2, \dots, x_n) = \bar{x}_i f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) + x_i f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$$

$$f(x_1, x_2, \dots, x_n) = [\bar{x}_i + f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)] [x_i + f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)]$$

- Proof:

$$\begin{aligned} x_i = 0 \Rightarrow f(x_1, x_2, \dots, x_n) &= 1 \cdot f(x_1, \dots, 0, \dots, x_n) + 0 \cdot f(x_1, \dots, 1, \dots, x_n) = \\ &= f(x_1, \dots, 0, \dots, x_n) \end{aligned}$$

$$\begin{aligned} x_i = 1 \Rightarrow f(x_1, x_2, \dots, x_n) &= 0 \cdot f(x_1, \dots, 0, \dots, x_n) + 1 \cdot f(x_1, \dots, 1, \dots, x_n) = \\ &= f(x_1, \dots, 1, \dots, x_n) \end{aligned}$$

- The proof of the other form is given by duality

Corollary of Shannon's Expansion Theorem

- Apply the Theorem recursively:

$$\begin{aligned}
 f(a,b,c) &= \bar{a}f(0,b,c) + af(1,b,c) = \\
 &= \bar{a}(\bar{b}f(0,0,c) + bf(0,1,c)) + a(\bar{b}f(1,0,c) + bf(1,1,c)) = \\
 &= \bar{a}\bar{b}f(0,0,c) + \bar{a}bf(0,1,c) + a\bar{b}f(1,0,c) + abf(1,1,c) = \\
 &= \bar{a}\bar{b}\bar{c}f(0,0,0) + \bar{a}\bar{b}cf(0,0,1) + \bar{a}\bar{b}cf(0,1,0) + \bar{a}bcf(0,1,1) + \\
 &\quad + \bar{a}\bar{b}c\bar{f}(1,0,0) + \bar{a}\bar{b}cf(1,0,1) + a\bar{b}\bar{c}f(1,1,0) + abc\bar{f}(1,1,1) = \\
 &= \sum_3 m_i k_i
 \end{aligned}$$

- A function can be expressed as the sum of all minterms (m_i) multiplied by a coefficient (k_i) equal to the value of the function when each variable is substituted by 0, if the variable appears as inverted in the minterm, or 1 otherwise.

First canonical form

- A function can be expressed as the sum of all minterms for which the function evaluates to 1

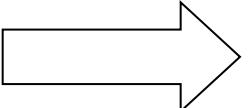
a	b	c	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

$$\begin{aligned}
 f(a,b,c) &= \sum_3(0,2,5) = \sum_3 m(0,2,5) = \\
 &= \bar{a}\bar{b}\bar{c} + \bar{a}b\bar{c} + a\bar{b}\bar{c}
 \end{aligned}$$

Second canonical form

- A function can be expressed as the product of all maxterms for which the function evaluates to 0

a	b	c	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0



$$\begin{aligned}
 f(a,b,c) &= \prod_3 (1,3,4,6,7) = \prod_3 M(1,3,4,6,7) = \\
 &= (a + b + \bar{c})(a + \bar{b} + \bar{c})(\bar{a} + b + c) \\
 &\quad (\bar{a} + \bar{b} + c)(\bar{a} + \bar{b} + \bar{c})
 \end{aligned}$$

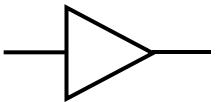
ATTENTION: minterms use the opposite rule!

Logic gates

- Logic gates are electronic circuits that implement the basic functions of Boolean Algebra
- There is a symbol for each gate
- Identity

$$z = a$$

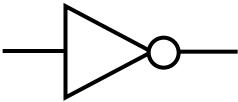
a	a
0	0
0	1



- NOT gate or inverter

$$z = \bar{a}$$

a	\bar{a}
0	1
1	0



AND and OR gates

- AND gate

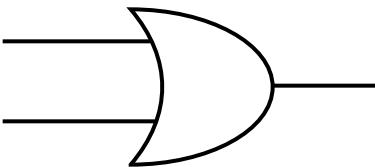
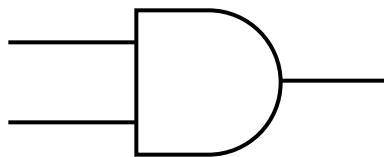
$$z = a \cdot b$$

a	b	$a \cdot b$
0	0	0
0	1	0
1	0	0
1	1	1

- OR gate

$$z = a + b$$

a	b	$a + b$
0	0	0
0	1	1
1	0	1
1	1	1



NAND and NOR gates

- NAND gate

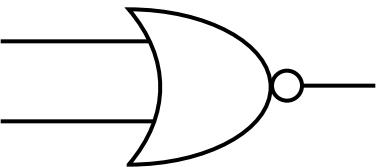
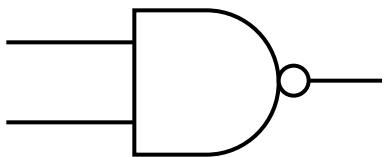
$$z = \overline{a \cdot b} = \overline{a} + \overline{b}$$

a	b	$\overline{a \cdot b}$
0	0	1
0	1	1
1	0	1
1	1	0

- NOR gate

$$z = \overline{a + b} = \overline{a} \overline{b}$$

a	b	$\overline{a + b}$
0	0	1
0	1	0
1	0	0
1	1	0



XOR and XNOR gates

- XOR (Exclusive-OR) gate

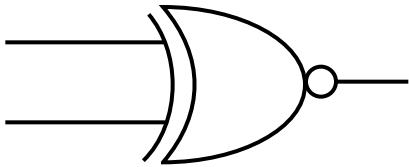
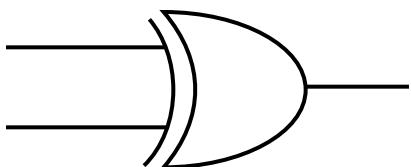
$$z = a \oplus b = \bar{a}\bar{b} + a\bar{b} = (\bar{a} + \bar{b})(a + b)$$

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

- XNOR (Exclusive-NOR) gate

$$z = \overline{a \oplus b} = ab + \bar{a}\bar{b} = (\bar{a} + b)(a + \bar{b})$$

a	b	$\overline{a \oplus b}$
0	0	1
0	1	0
1	0	0
1	1	1

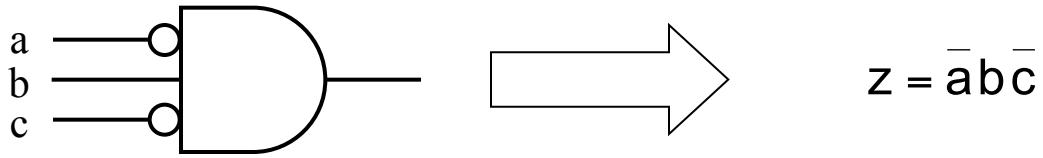


Generalization to n inputs

Gate	Output value	
	0	1
AND	Any input = 0	All inputs = 1
OR	All inputs = 0	Any input = 1
NAND	All inputs = 1	Any input = 0
NOR	Any input = 1	All inputs = 0
XOR	There is an even number of inputs = 1	There is an odd number of inputs = 1
XNOR	There is an odd number of inputs = 1	There is an even number of inputs = 1

Other symbols

- A circle at an input or output indicates inversion



Digital technologies

- Logic gates are electronic circuits
- Logic levels (0 or 1) are represented by means of a voltage level
- “Positive logic” is generally used
 - High voltage (5V, 3.3V, 2.5 V, etc.) → 1
 - Low voltage (0V) → 0
- There are many technologies, depending on the way logic gates are implemented and the features obtained by every implementation

Logic families

- The set of basic digital components, such as logic gates and others that will be studied along the course, is commonly known as *74 Series of Family*
- There are many subfamilies:
 - According to the operating temperature range:
 - 74 Series : 0° a 70°
 - 54 Series : -55° a 125°
 - According to technology:
 - LS
 - ALS
 - F
 - HC
 - AHC
 - G
 -

Logic families

- Component denomination:
 - <Series><Subfamily><Component>
- Example: 74HC00
 - 74 Series: conventional temperature range
 - HC subfamily (High speed CMOS)
 - Component 00: 4 NAND gates, 2 inputs
- Important: members of different subfamilies are not compatible
 - Components of different subfamilies should not be assembled together in a circuit

Datasheets



June 1989

54LS00/DM54LS00/DM74LS00 Quad 2-Input NAND Gates

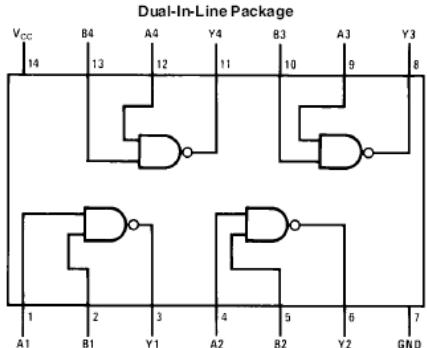
General Description

This device contains four independent gates each of which performs the logic NAND function.

Features

- Alternate Military/Aerospace device (54LS00) is available. Contact a National Semiconductor Sales Office/Distributor for specifications.

Connection Diagram



TL/F/8439-1

Order Number 54LS00DMQB, 54LS00FMQB, 54LS00LMQB, DM54LS00J, DM54LS00W, DM74LS00M or DM74LS00N
See NS Package Number E20A, J14A, M14A, N14A or W14B

54LS00/DM54LS00/DM74LS00 Quad 2-Input NAND Gates

Function Table

$$Y = \overline{AB}$$

Inputs		Output
A	B	Y
L	L	H
L	H	H
H	L	H
H	H	L

H = High Logic Level

L = Low Logic Level

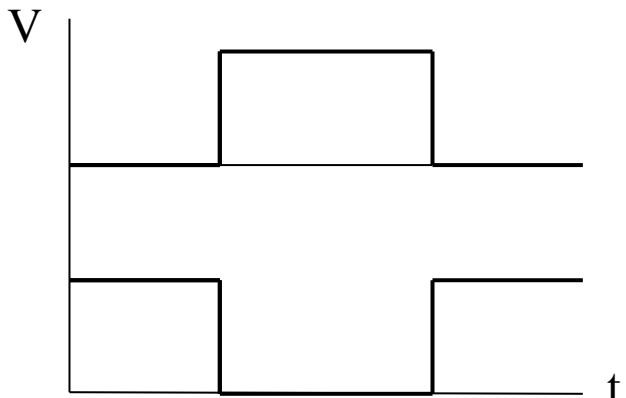
Features of digital technologies

- Main features:
 - Operating temperature range
 - Voltage supply
 - Noise margin (voltage intervals associated to each logic value)
 - Switching delay
 - Power
 - Others
- Each technology or subfamily shows different values for these features

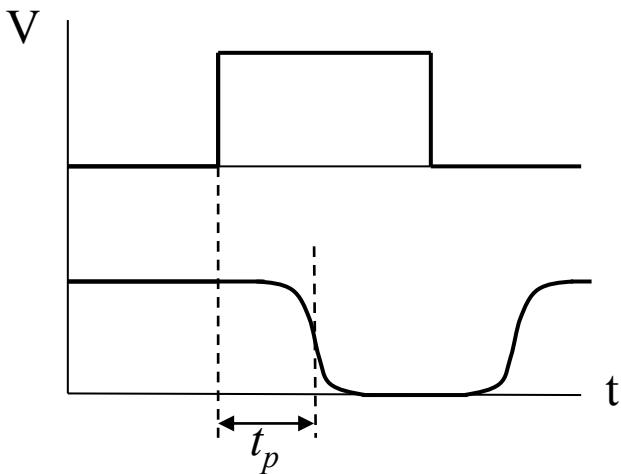
Delays

- Logic gates do not switch immediately

Ideal inverter



Real inverter



- Delay limits the operating speed of circuits

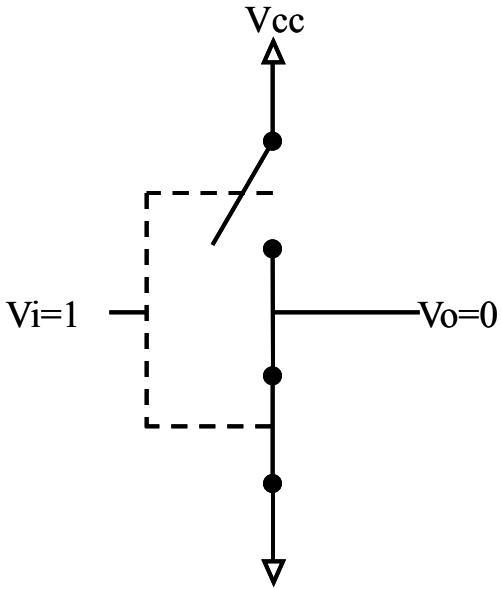
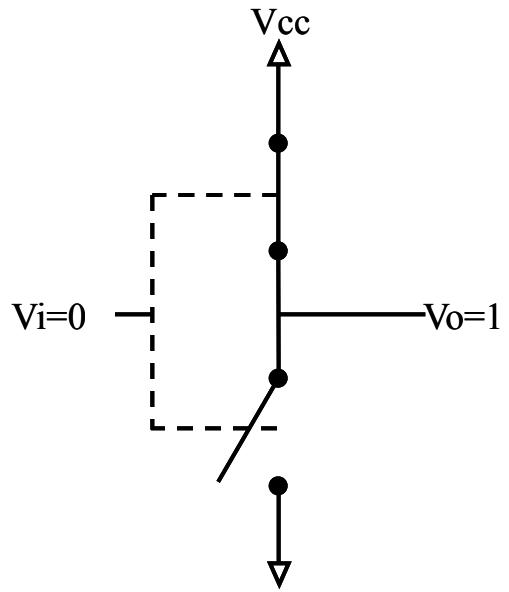
Power

- Logic gates consume energy:
 - Static power: power taken when the circuit is on, without logic activity
 - Dynamic power: power taken when logic gates commute
- In CMOS technology (currently the most widely used), static power consumption is very small. However,
 - Modern circuits can have more than 10^8 logic gates!
 - Dynamic power consumption is proportional to operating frequency
- Power consumption is an important problem:
 - Consumed energy is transformed into heat, which must be dissipated. Dissipation can be difficult if the circuit has a large power consumption
 - In portable devices, battery size and weight are at a premium

CMOS technology

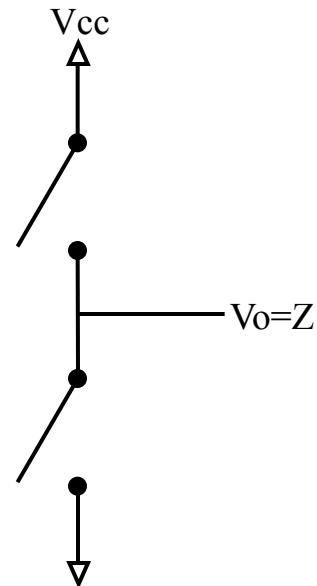
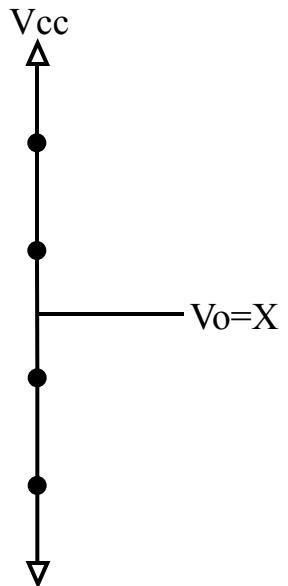
- CMOS (Complementary Metal Oxide Semiconductor) technology is the most widespread technology today
- Based upon:
 - MOS transistors: voltage controlled switches
 - Complementary: every transistor or switch has a complementary one. When a switch is open, its complementary switch is closed and viceversa

CMOS Inverter



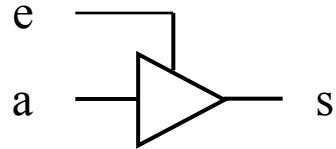
Metalogic values

- There are situations that do not correspond to any logic value
 - Shortcircuit (X)
 - High impedance or tristate (Z)



Tristate buffer

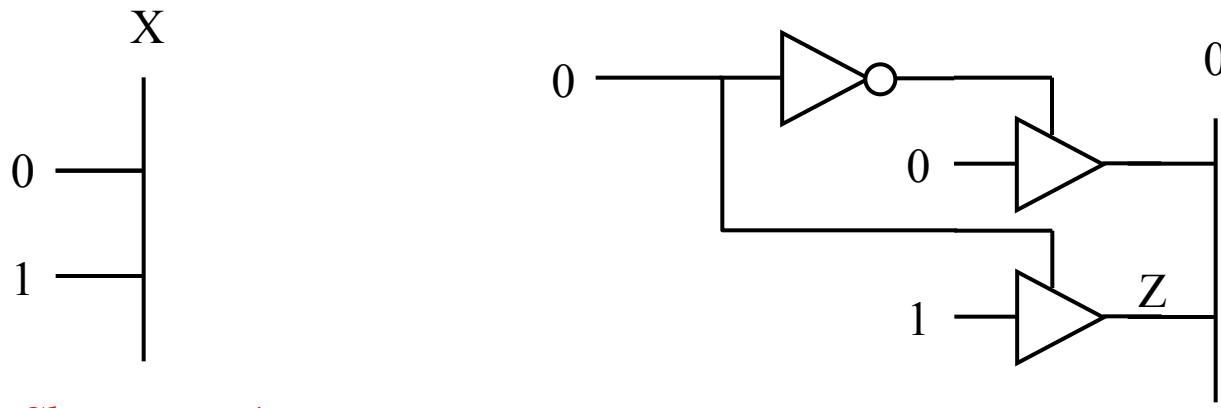
- A special sort of logic gate that can put its output in high impedance



e	a	s
0	0	Z
0	1	Z
1	0	0
1	1	1

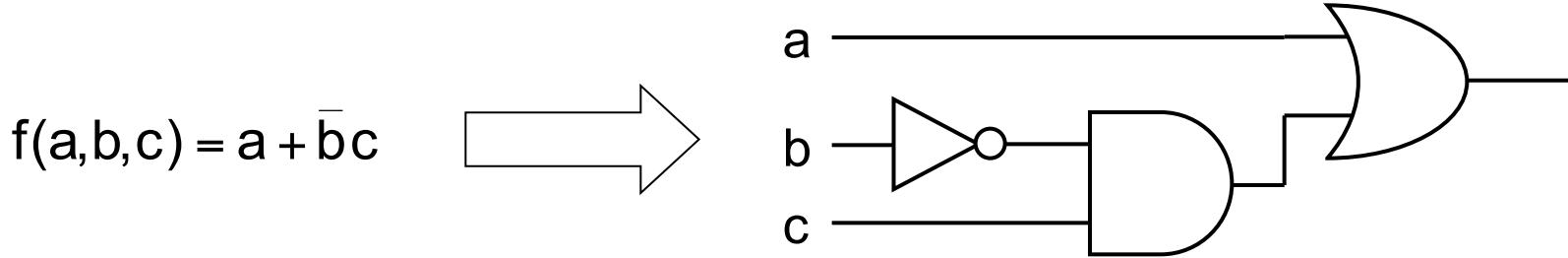
Tristate buffer

- Tristate buffers are useful to avoid shortcircuits when connecting multiple signals to a common point



Implementation of a logic function with logic gates

- Take the function expression and substitute logic operations by logic gates
- Example:



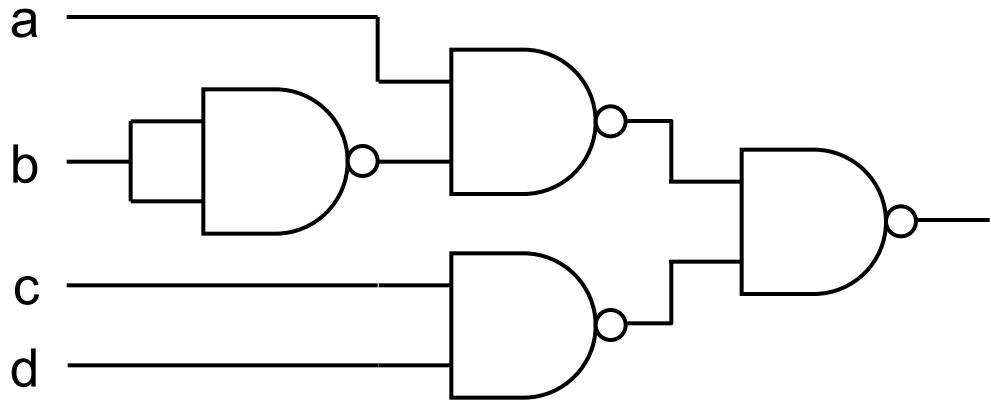
Funtionally complete operations

- A set of functions is said to be functionally complete (or universal) if and only if every logic function can be expressed entirely by means of operations of this set
 - {AND} is *not* a complete set
 - {AND, NOT} is a complete set
 - {OR, NOT} is a complete set
 - {NAND} is a complete set
 - {NOR} is a complete set
- {NAND} and {NOR} sets are very useful, since they allow implementing any logic function with a single type of logic gates

Circuit implementation with NAND gates

- Direct application of De Morgan laws
- Example: $f(a,b,c) = a\bar{b} + c\bar{d} =$

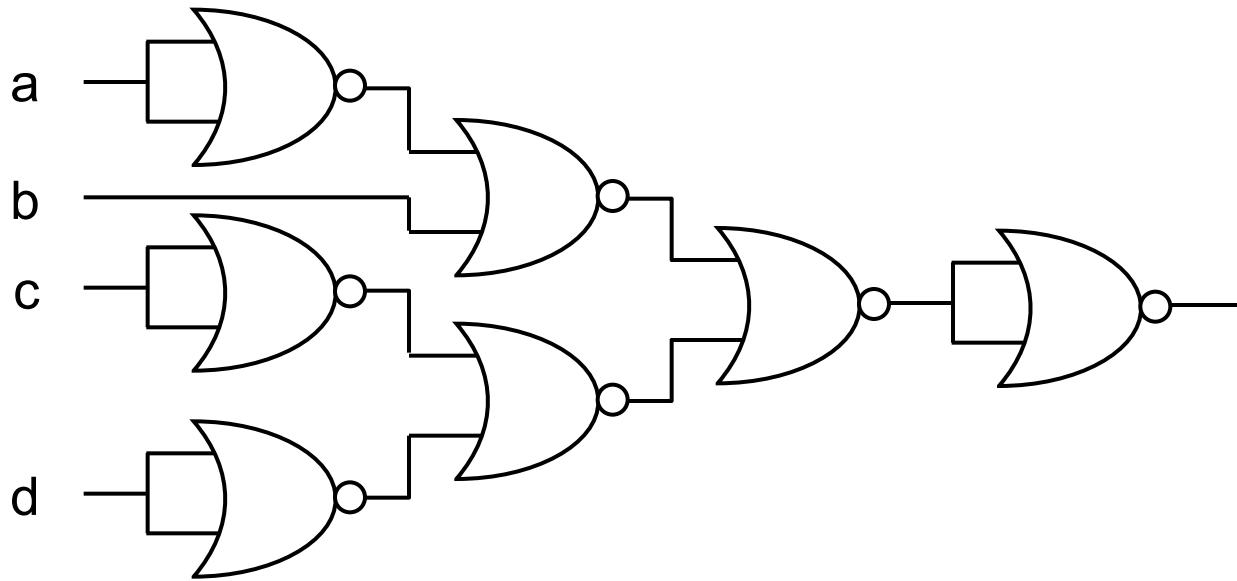
$$= \overline{\overline{a}\overline{b}} + \overline{c}\overline{d} = \overline{a}\overline{b} \cdot \overline{c}\overline{d}$$



Circuit implementation with NOR gates

- Direct application of De Morgan laws
- Example: $f(a,b,c) = a\bar{b} + c\bar{d} =$

$$= \overline{\overline{ab}} + \overline{\overline{cd}} = \overline{a+b+c+d}$$



Minimization of logic functions

- A logic function has many equivalent expressions
 - The simplest expression will lead to a better implementation
- Optimization criteria:
 - Area or size:
 - Smaller logic gate count
 - Logic gates with less inputs
 - Speed or delay:
 - Smaller number of logic gates from input to output
- We will focus on area optimization

Minimization of logic functions

- Optimization methods
 - Manual: direct application of Boolean Algebra laws
 - Very difficult, unsystematic
 - Two level: find an optimal expression as a sum of products or a product of sums
 - There are systematic optimal solutions
 - Feasible in a manual way (for few variables) or with the help of a computer
 - Multilevel
 - Better solution, but much more difficult
 - Feasible only with the help of a computer

Karnaugh map method

- Two level optimization method
- Can be applied in a manual way up to 6 variables
- Based on the Adjacency Property
 - $\forall E, x \in B \Rightarrow E x + E \bar{x} = E(x + \bar{x}) = E$
 $(E + x)(E + \bar{x}) = E + (x \cdot \bar{x}) = E \quad (\text{dual})$
 - Two terms are adjacent if they are identical except for one literal, that appears inverted in one term and non-inverted in the other term
 - The two terms are simplified into a single one by eliminating the differentiating literal

Applying the Adjacency Property

- Example:

$$f(a,b,c) = \sum_3 (0,1,2,3,7) = \overbrace{\bar{a}\bar{b}\bar{c}} + \overbrace{\bar{a}\bar{b}c} + \overbrace{\bar{a}b\bar{c}} + \overbrace{\bar{a}bc} + abc =$$

$$\begin{aligned}
 &= \overbrace{\bar{a}\bar{b}} + \overbrace{\bar{a}b} + \overbrace{bc} \\
 &= \overbrace{\bar{a}} + \overbrace{bc}
 \end{aligned}$$

The diagram shows the simplification of a Boolean expression. It starts with five terms in a sum: $\bar{a}\bar{b}\bar{c}$, $\bar{a}\bar{b}c$, $\bar{a}b\bar{c}$, $\bar{a}bc$, and abc . Braces group terms with common factors: the first two terms share $\bar{a}\bar{b}$, the last three share \bar{a} , and the last term abc is grouped separately. Arrows point from each brace to its respective factor in the terms. The expression then simplifies to $\bar{a} + bc$.

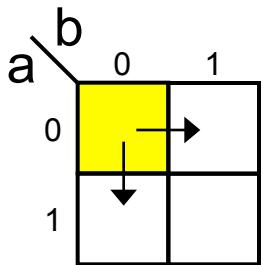
- Realizing adjacencies may be difficult in practice

Karnaugh map

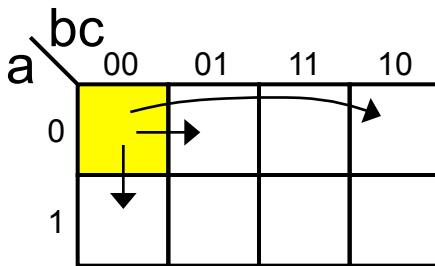
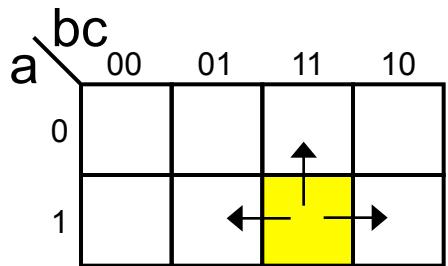
- A map that presents the truth table of a function in such a way that adjacent terms are contiguous :
 - There is a cell for each combination or term
 - Cells are numbered in Gray code
 - In a map for n variables, each cell has n adjacent cells that correspond to the combinations that result from inverting each of the n variables

Karnaugh maps: adjacencies

- Two variables

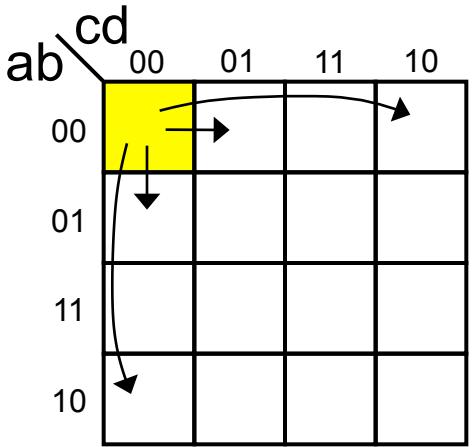
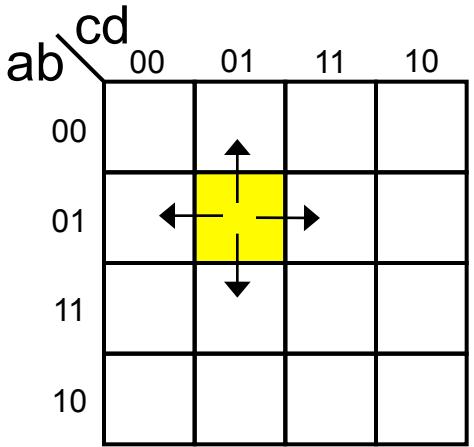


- Three variables



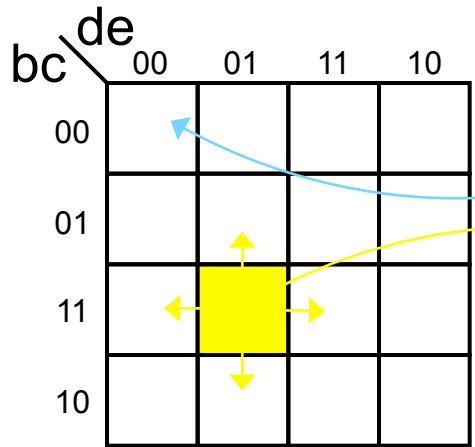
Karnaugh maps: adjacencies

- Four variables

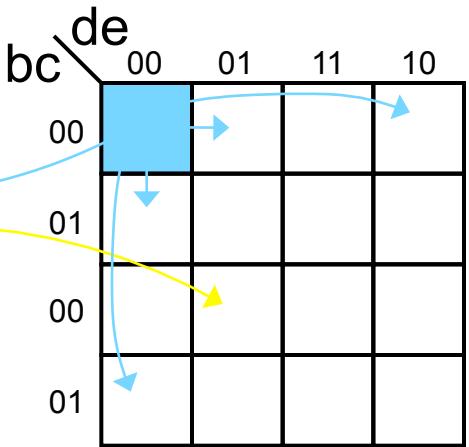


Karnaugh maps: adjacencies

- Five variables



$a = 0$



$a = 1$

Karnaugh map method: cell numbering

- Two variables

	$a\backslash b$	0	1
0	0	0	1
1	2	3	

- Four variables

	$ab\backslash cd$	00	01	11	10
00	0	1	3	2	
01	4	5	7	6	
11	12	13	15	14	
10	8	9	11	10	

- Three variables

	$a\backslash bc$	00	01	11	10
0	0	1	3	2	
1	4	5	7	6	

Karnaugh map method: cell numbering

- Five variables

	$bc\backslash de$	00	01	11	10
bc	00	0	1	3	2
	01	4	5	7	6
	11	12	13	15	14
	10	8	9	11	10

$a = 0$

	$bc\backslash de$	00	01	11	10
bc	00	16	17	19	18
	01	20	21	23	22
	00	28	29	31	30
	01	24	25	27	26

$a = 1$

Function representation in a Karnaugh map

- Cells corresponding to the minterms or the maxterms of the function are marked by a 1 or a 0, respectively
- Example:

$$\begin{aligned}
 f(a,b,c) &= \sum_3 (0,1,2,3,7) = \\
 &= \prod_3 (4,5,6)
 \end{aligned}$$



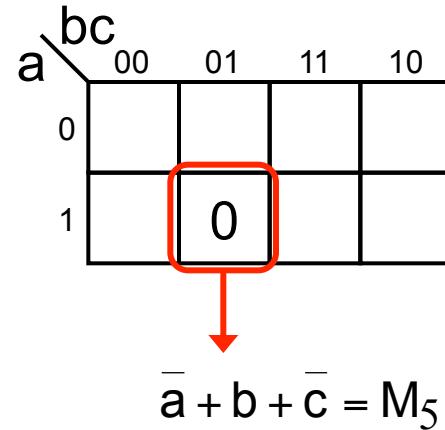
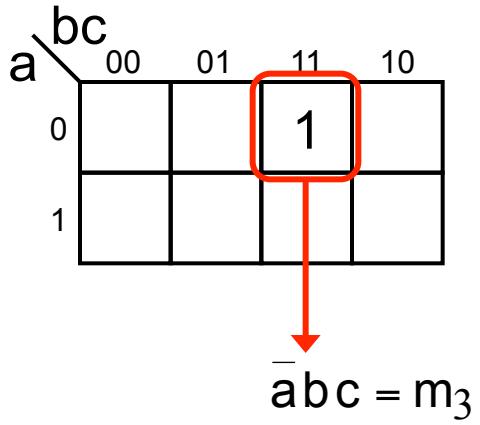
	<i>a</i>	<i>bc</i>	00	01	11	10
0			1	1	1	1
1					1	



	<i>a</i>	<i>bc</i>	00	01	11	10
0						
1			0	0		0

Deriving an expression from a Karnaugh map

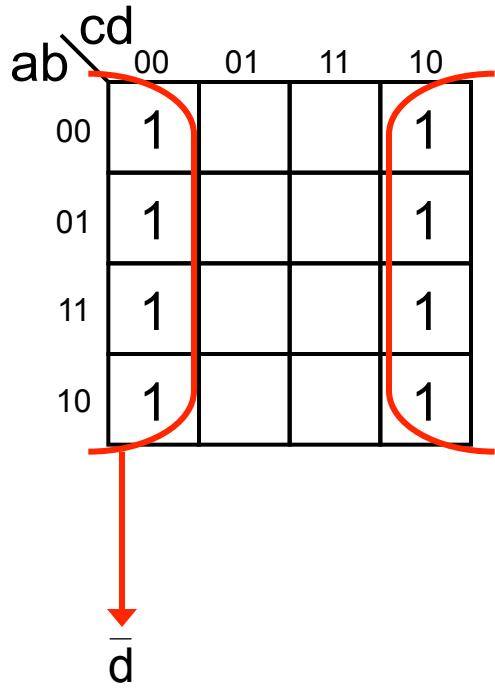
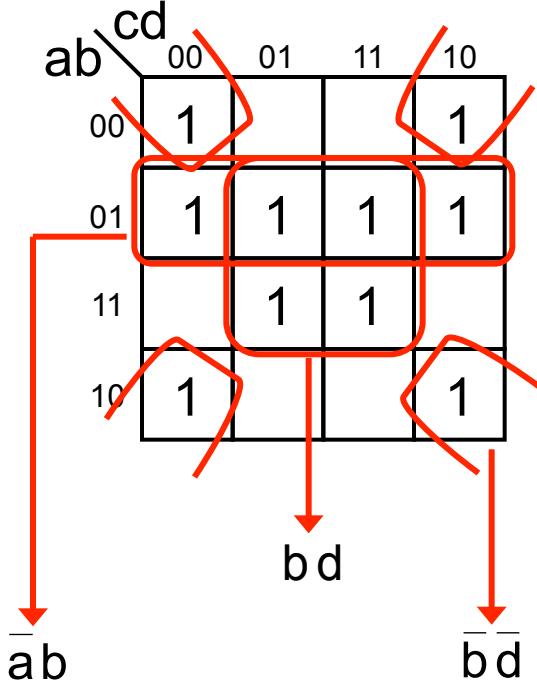
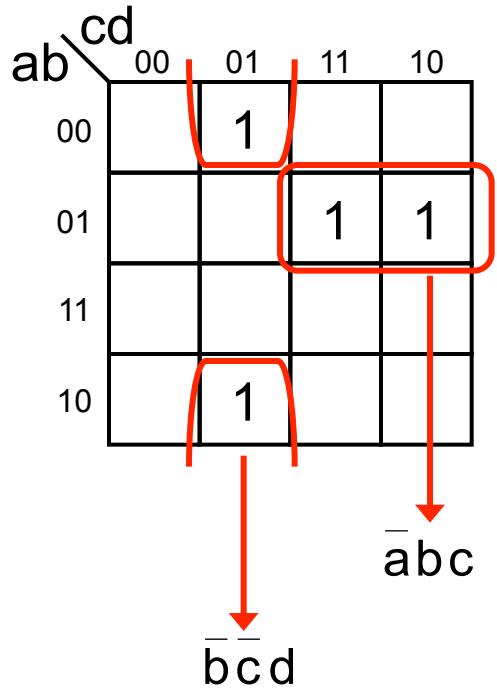
- Follow the rules for minterms and maxterms
 - Minterms rule
 - 0 → inverted variable
 - 1 → non-inverted variable
 - Maxterms rule
 - 0 → non-inverted variable
 - 1 → inverted variable



Simplification with Karnaugh maps

- Two options
 - By minterms (ones): result is a sum of products
 - By maxterms (zeros): result is product of sums
- Search for groups of adjacent cells
 - A group of 2 adjacent cells removes 1 variable
 - A group of 4 adjacent cells removes 2 variables
 - A group of 8 adjacent cells removes 3 variables
 - A group of 16 adjacent cells removes 4 variables
 -
- Goal: cover all minterms (maxterms) with the largest possible groups and the smaller number of groups
 - Terms can be covered more than once, if needed (absorption property)

Simplification: groups

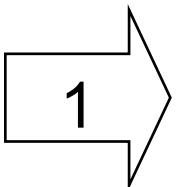


Simplification with Karnaugh maps: Algorithm

- Systematic algorithm
 1. Cover cells that cannot form groups of 2 cells
 2. Cover cells that can form groups of 2 cells, but not of 4 cells
 3. Cover cells that can form groups of 4 cells, but not of 8 cells
 4. Cover cells that can form groups of 8 cells, but not of 16 cells
 5. ...
- If there is more than one choice at any step:
 - First cover cells with less choices

Simplification with Karnaugh maps: Example

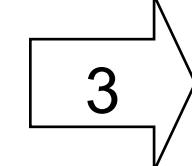
	$ab \backslash cd$	00	01	11	10
00		1			
01			1	1	
11			1	1	
10			1		



	$ab \backslash cd$	00	01	11	10
00		1			
01			1	1	
11			1	1	
10			1		



	$ab \backslash cd$	00	01	11	10
00		1			
01			1	1	
11			1	1	
10			1		



	$ab \backslash cd$	00	01	11	10
00		1			
01			1	1	
11			1	1	
10			1		

Incompletely specified functions

- An incompletely specified (or incomplete) function is a function that is not specified for some combination of input values
- Incomplete functions occur in practice:
 - When inputs come from another circuit that cannot produce some particular output combinations by construction
 - When there are combination for which the function value is meaningless or don't-care
- Notation:
 - A don't-care is represented as 'X' or '-'
 - The don't-care set is typically represented by Δ

Incompletely specified functions

- Example: Function to determine if a BCD number is even
 - Codes from 10 to 15 are meaningless in BCD

$$\begin{aligned}
 f(b_3, b_2, b_1, b_0) &= \sum_4 (1, 3, 5, 7, 9) + \Delta(10, 11, 12, 13, 14, 15) = \\
 &= \prod_4 (0, 2, 4, 6, 8) + \Delta(10, 11, 12, 13, 14, 15)
 \end{aligned}$$

Don't-care terms

b3	b2	b1	b0	f
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

Minimization of incompletely specified functions

- Don't-care terms are “wild cards”: they can be covered or not, as needed to form larger groups

	$b_1 b_0$	00	01	11	10
$b_3 b_2$	00	1	1		
	01	1	1		
	11	X	X	X	X
	10	1	X	X	X

$$f(b_3, b_2, b_1, b_0) = \overline{b_3} b_0 + \overline{b_2} \overline{b_1} b_0$$

	$b_1 b_0$	00	01	11	10
$b_3 b_2$	00	1	1		
	01	1	1		
	11	X	X	X	X
	10	1	X	X	X

$$f(b_3, b_2, b_1, b_0) = b_0$$

Correct

Multiple functions

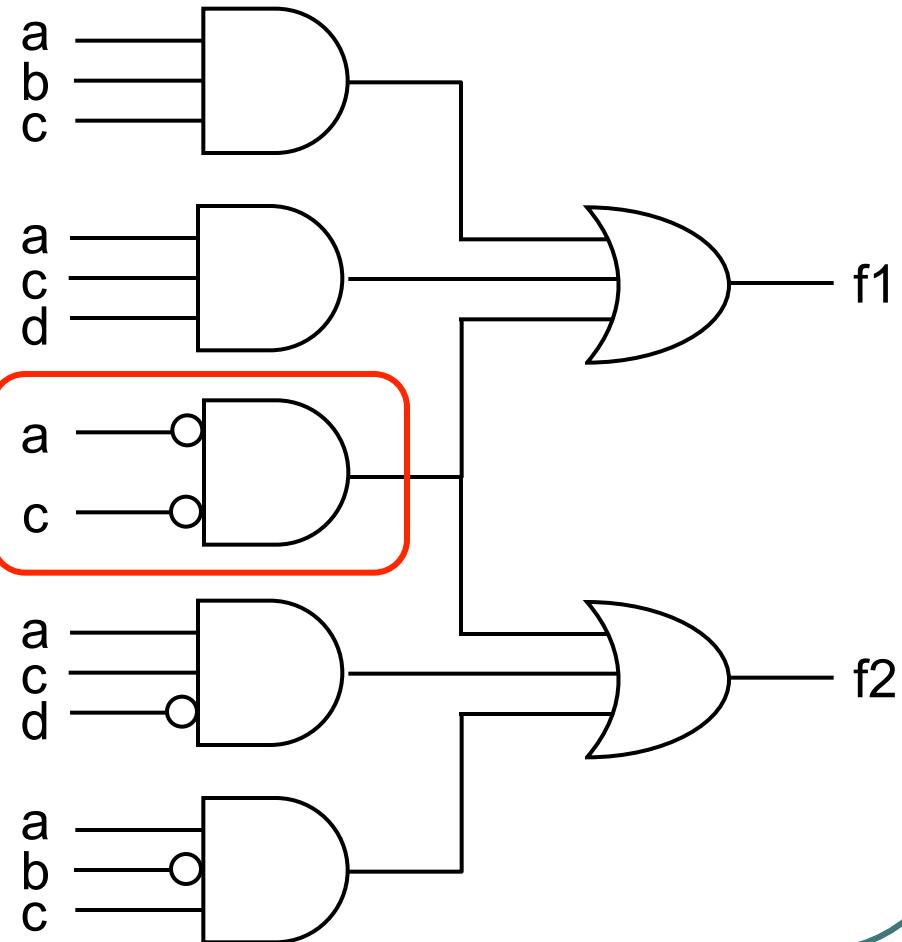
- Digital circuits generally implement multiple functions: several functions or multiple-output functions
- Multiple function can be better implemented by considering them jointly
 - Common parts or terms can be shared to save logic
- Decomposition of multiple functions in order to maximize common subexpressions is difficult
 - Algorithms are difficult to apply in a manual way
 - We will generally do it by visual inspection

Multiple functions: Example

$$f_1(a,b,c,d) = \overline{a}\overline{c} + abc + acd$$

$$f_2(a,b,c,d) = \overline{a}\overline{c} + a\overline{b}c + a\overline{c}d$$

Common terms



Multiple functions: Example

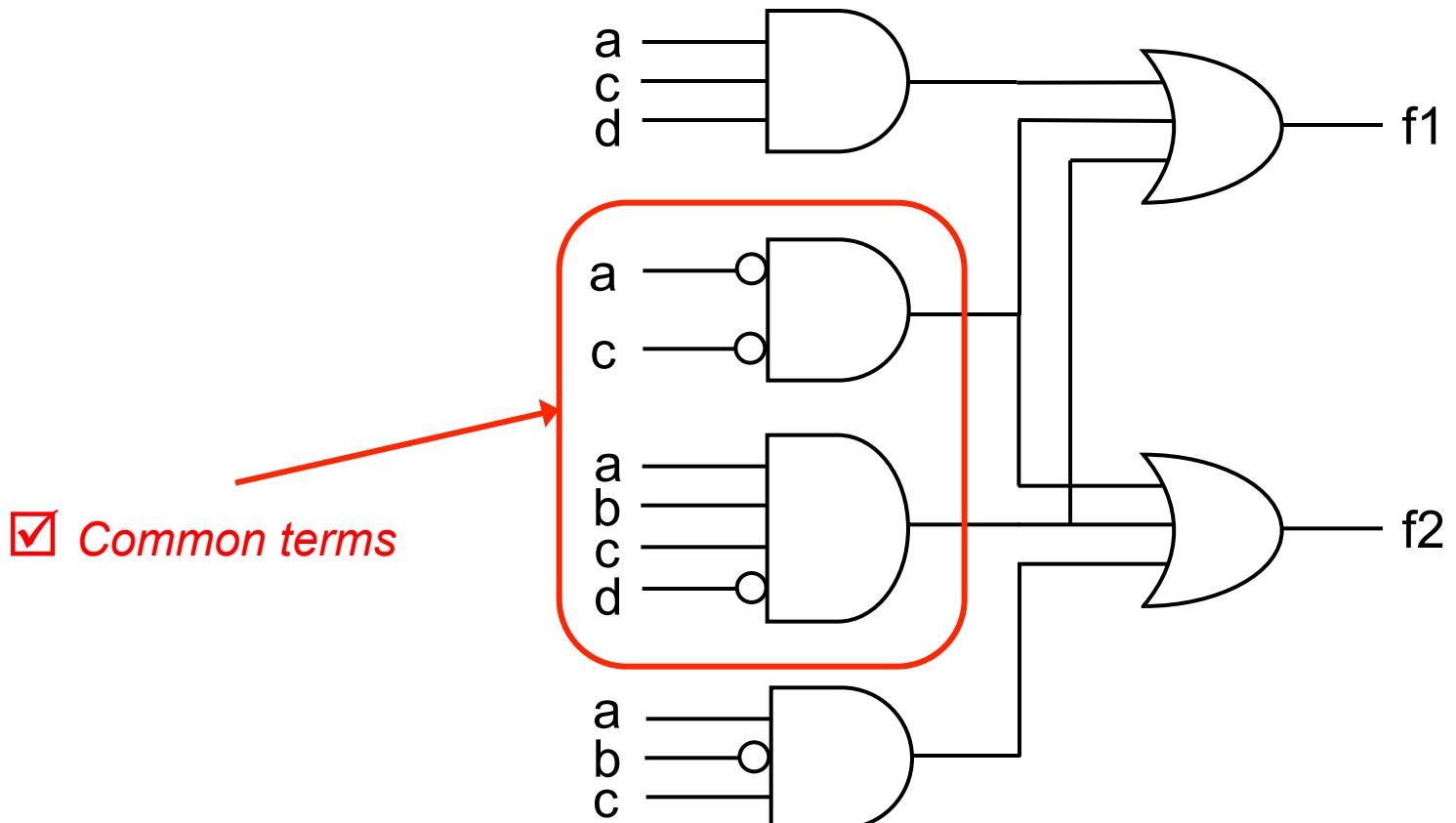
- More common terms may be found

$$f_1(a,b,c,d) = \overline{\overline{a}}\overline{c} + abc + acd = \overline{\overline{a}}\overline{c} + \overline{a}\overline{b}cd + acd$$

$$f_2(a,b,c,d) = \overline{\overline{a}}\overline{c} + \overline{a}\overline{b}c + ac\overline{d} = \overline{\overline{a}}\overline{c} + \overline{a}\overline{b}cd + \overline{a}\overline{b}c$$

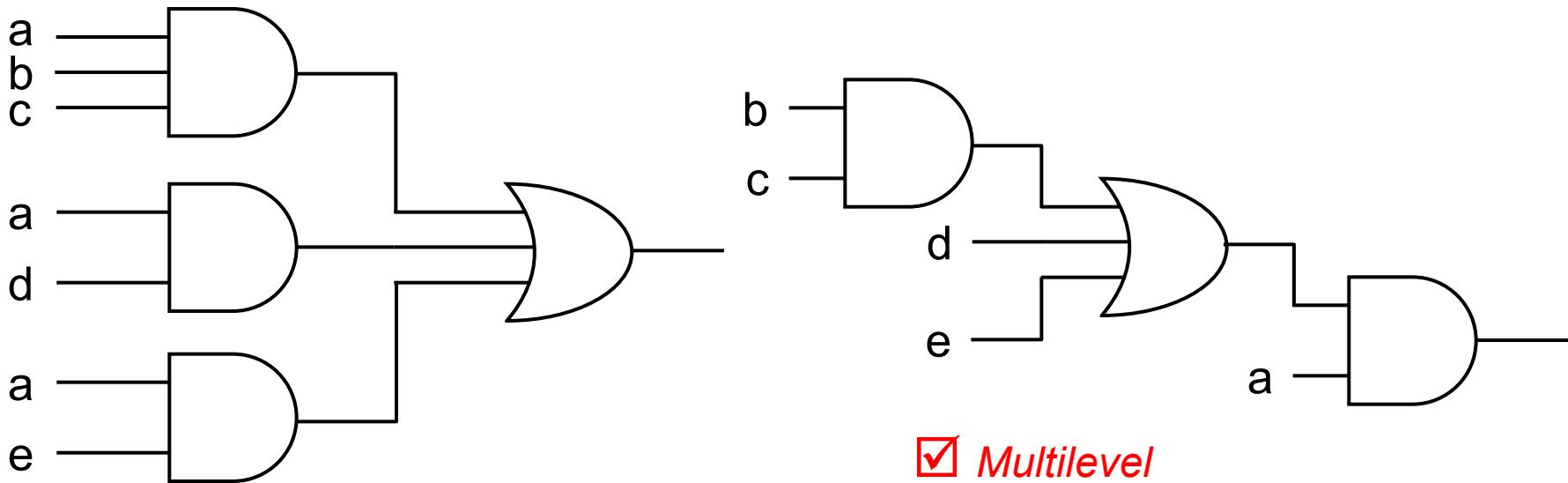
- Function expressions are not optimal separately, but they are optimal as a group!
- Design tools include algorithms to minimize multiple functions

Multiple functions: Example



Multilevel synthesis

- Better solutions are possible by removing the two-level constraint
 - Computer-aided heuristic algorithms are used
- Example: $f(a,b,c,d,e) = abc + ad + ae = a(bc + d + e)$



Multilevel

Optimization tools

- Manual methods:
 - Only two-level, few variables
- Software tools
 - Multilevel, multiple functions, many variables
 - Area and delay optimization
 - Generally incorporated into logic synthesis tools
- Logic synthesis tools
 - Work as a compiler, from a design description in schematic form or in a Hardware Description Language
 - Optimize the design and generate the logic gate description for the selected technology



References

- “Introduction to Digital Logic Design”. J. P. Hayes. Ed. Addison-Wesley
- “Circuitos y sistemas digitales”. J. E. García Sánchez, D. G. Tomás, M. Martínez Iniesta. Ed. Tebar-Flores