Universidad
Carlos III de Madrid
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Departamento de Mecánica de Medios Continuos y Teoría de Estructuras

Master en Mecánica Estructural Avanzada

Mecánica de Materiales Compuestos

Tema 2. Análisis de la lámina

Curso 2010/2011

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Tema 2.2
Elasticidad anisótropa aplicada a la lámina

- Matriz de rigidez en ejes locales
- Matriz de rigidez en ejes globales
- Matrices de flexibilidad
- Constantes aparentes de la lámina

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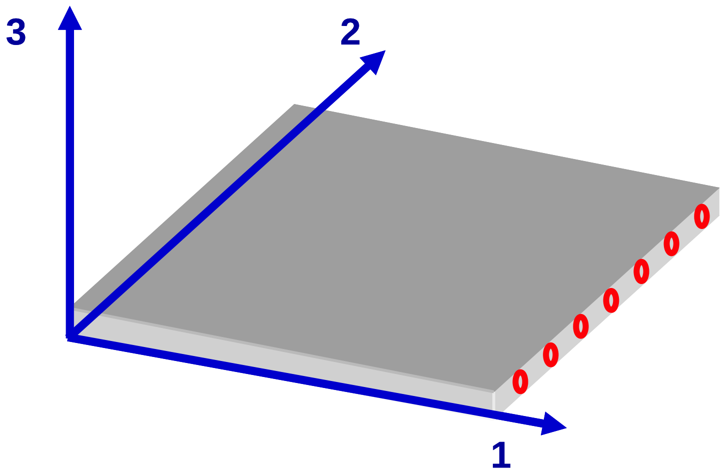
Matriz de rigidez en ejes locales



Estado de tensión plana

$$[Q]=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}$$

$$[Q]=\begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\cdot\nu_{21}} & \dfrac{\nu_{21}\cdot E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ \dfrac{\nu_{12}\cdot E_1}{1-\nu_{12}\cdot\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\cdot\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

$$\sigma_3=0 \qquad \tau_{13}=0 \qquad \tau_{23}=0$$

$$\longrightarrow \quad [T]=\begin{bmatrix} \sigma_1 & \tau_{12} \\ \tau_{12} & \sigma_2 \end{bmatrix} \quad \longrightarrow \quad \{\sigma\}=\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

$$\{\varepsilon\}=\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$ ¡!

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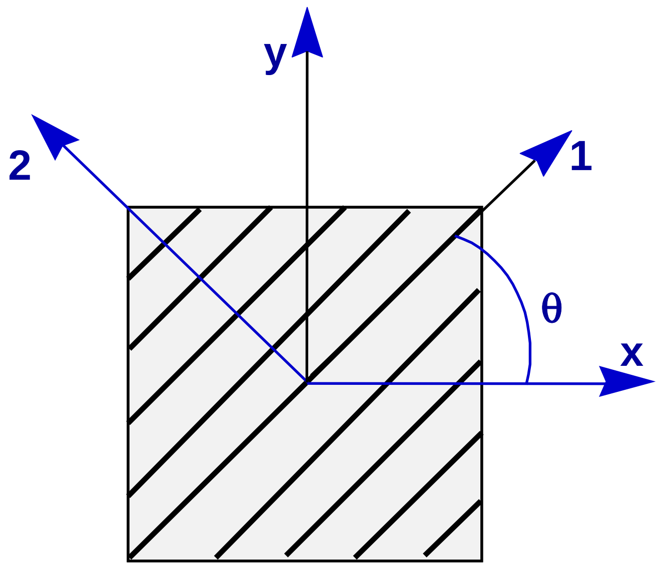
Matriz de rigidez en ejes globales



En ejes 12: $\{\sigma\}^{12} = [Q] \cdot \{\varepsilon\}^{12}$

En ejes xy: $\{\sigma\}^{xy} = [\overline{Q}] \cdot \{\varepsilon\}^{xy}$

$$[Q] \Rightarrow [\overline{Q}]$$

¿?

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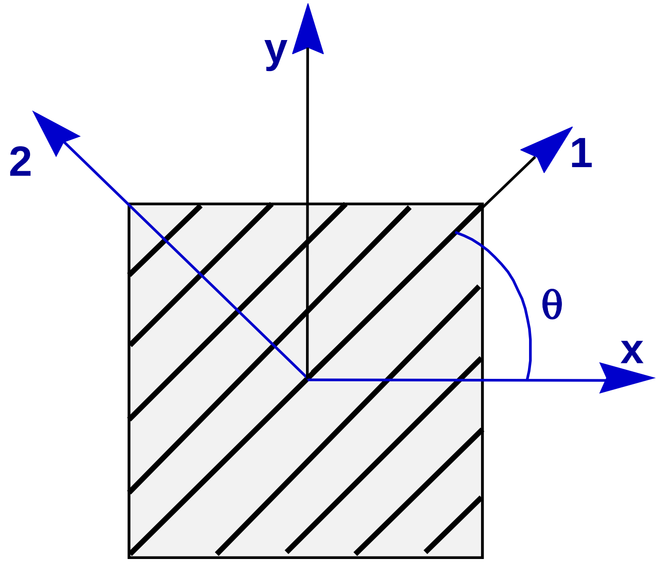
Matriz de rigidez en ejes globales



$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{matrix}\right\} = [T]\cdot\left\{\begin{matrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\cdot\gamma_{12}\end{matrix}\right\} = [T]\cdot\left\{\begin{matrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\cdot\gamma_{xy}\end{matrix}\right\}$$

$m = \cos\theta$ $\qquad$ $n = \mathrm{sen}\theta$

$$[T] = \begin{bmatrix} m^2 & n^2 & 2\cdot m\cdot n \\ n^2 & m^2 & -2\cdot m\cdot n \\ -m\cdot n & m\cdot n & (m^2 - n^2) \end{bmatrix}$$

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Matriz de rigidez en ejes globales

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [Q]\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = [\overline{Q}]\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = [R]^{-1}\cdot\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \frac{1}{2}\cdot\gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix} = [R]^{-1}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \frac{1}{2}\cdot\gamma_{12}\end{Bmatrix}$$

$$[\mathrm{R}] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$$

$$[\mathrm{R}]^{-1} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

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Matriz de rigidez en ejes globales

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \cdot \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

→

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

→

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R]^{-1} \cdot \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2} \cdot \gamma_{12} \end{Bmatrix}$$

→

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2} \cdot \gamma_{12} \end{Bmatrix} = [T] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2} \cdot \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \cdot [Q] \cdot [R]^{-1} \cdot [T] \cdot [R] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

→

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{xx} & \overline{Q}_{xy} & \overline{Q}_{xs} \\ \overline{Q}_{xy} & \overline{Q}_{22} & \overline{Q}_{ys} \\ \overline{Q}_{xs} & \overline{Q}_{ys} & \overline{Q}_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

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Matriz de rigidez en ejes globales

$$\overline{Q}_{xx} = Q_{11} \cdot m^4 + 2 \cdot (Q_{12} + 2 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{22} \cdot n^4$$

$$\overline{Q}_{yx} = (Q_{11} + Q_{22} - 4 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{12} \cdot (n^4 + m^4)$$

$$\overline{Q}_{yy} = Q_{11} \cdot n^4 + 2 \cdot (Q_{12} + 2 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{22} \cdot m^4$$

$$\overline{Q}_{xS} = (Q_{11} - Q_{12} - 2 \cdot Q_{SS}) \cdot n \cdot m^3 + (Q_{12} - Q_{22} + 2 \cdot Q_{SS}) \cdot m^3 \cdot n$$

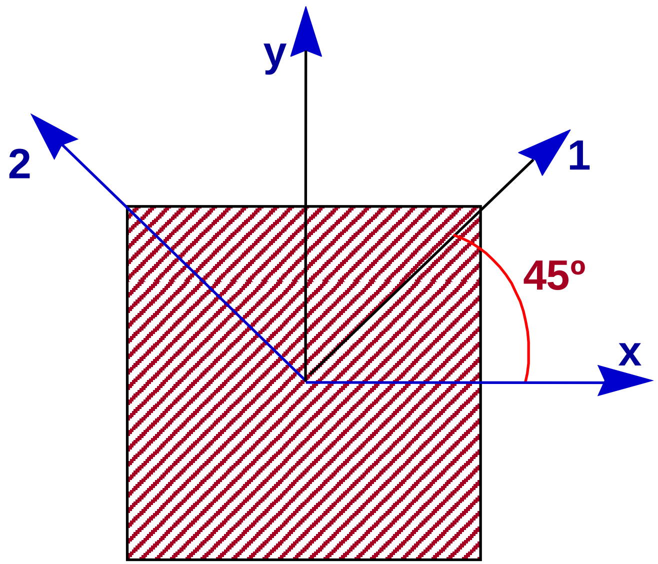
$$\overline{Q}_{yS} = (Q_{11} - Q_{12} - 2 \cdot Q_{SS}) \cdot n^3 \cdot m + (Q_{12} - Q_{22} + 2 \cdot Q_{SS}) \cdot n \cdot m^3$$

$$\overline{Q}_{SS} = (Q_{11} + Q_{22} - 2 \cdot Q_{12} - 2 \cdot Q_{SS}) \cdot n^2 \cdot m^2 + Q_{SS} \cdot (n^4 + m^4)$$

$m = \cos\theta \qquad n = \text{sen}\theta$

Matriz de rigidez en ejes globales

Matriz de rigidez en ejes globales de una lámina con fibras que forman ±45° con el eje X



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & Q_{22} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\bar{Q}_{xx} = \bar{Q}_{yy} = M + Q_{SS}$$

$$M = \frac{Q_{11} + Q_{22}}{4} + \frac{1}{2} \cdot Q_{12}$$

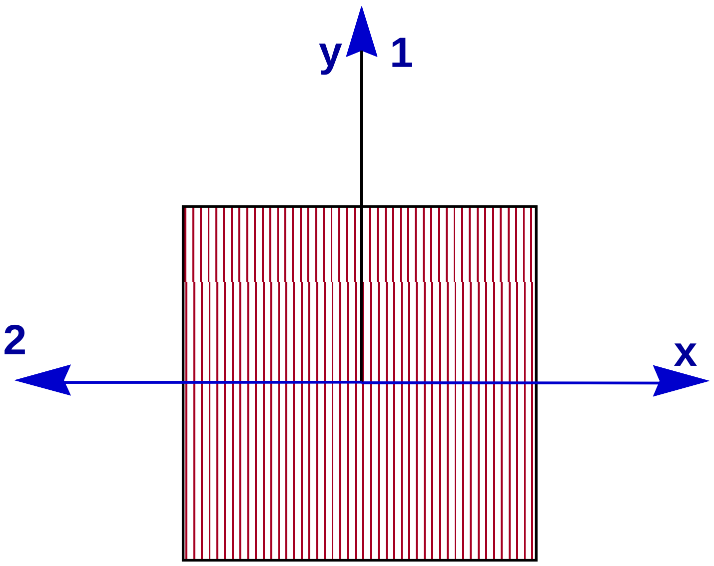
$$\bar{Q}_{SS} = M - \cdot Q_{12}$$

$$N = \frac{Q_{11} - Q_{22}}{4}$$

$$\bar{Q}_{xS} = \bar{Q}_{yS} = \pm N$$

$$\bar{Q}_{xy} = M - Q_{ss}$$

Matriz de rigidez en ejes globales de una lámina con fibras que forman ±90° con el eje X



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{xx} & \overline{Q}_{xy} & \overline{Q}_{xs} \\ \overline{Q}_{xy} & \overline{Q}_{yy} & \overline{Q}_{ys} \\ \overline{Q}_{xs} & \overline{Q}_{ys} & \overline{Q}_{ss} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\bar{Q}_{xx} = Q_{22} \qquad \bar{Q}_{xs} = 0$$

$$\bar{Q}_{yy} = Q_{11} \qquad \bar{Q}_{ys} = 0$$

$$\bar{Q}_{xy} = Q_{12} \qquad \bar{Q}_{ss} = Q_{ss}$$

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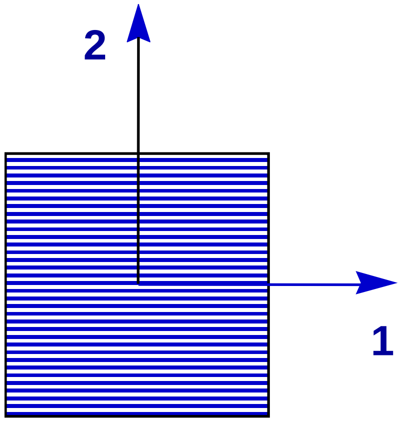
Constantes aparentes de una lámina



$$[Q]=\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{SS} \end{bmatrix}$$

⇓

$$Q_{11}=\frac{E_1}{1-\nu_{12}\cdot\nu_{21}}$$

$$Q_{22}=\frac{E_2}{1-\nu_{12}\cdot\nu_{21}}$$

$$Q_{12}=\frac{\nu_{21}\cdot E_2}{1-\nu_{12}\cdot\nu_{21}}$$

$$Q_{SS}=G_{12}$$

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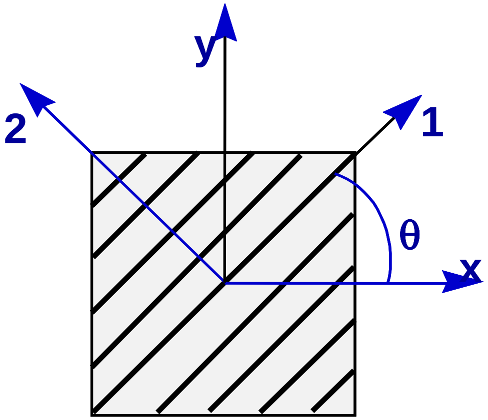
Constantes aparentes de una lámina



$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xS} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{yS} \\ \bar{Q}_{xS} & \bar{Q}_{yS} & \bar{Q}_{SS} \end{bmatrix}$$

⇩

$$\begin{cases} \bar{Q}_{xx} = \dfrac{E_x}{1 - \nu_{xy} \cdot \nu_{yx}} \\ \bar{Q}_{yy} = \dfrac{E_y}{1 - \nu_{xy} \cdot \nu_{yx}} \\ \bar{Q}_{xy} = \dfrac{\nu_{yx} \cdot E_y}{1 - \nu_{xy} \cdot \nu_{yx}} \\ \bar{Q}_{SS} = G_{xy} \end{cases}$$

⇦

$E_x \quad E_y \quad G_{xy} \quad \nu_{xy} \quad \nu_{yx}$

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Constantes aparentes de una lámina

$$\frac{1}{E_x} = \frac{1}{E_1} \cdot \cos^4\theta + \left(\frac{1}{G_{12}} - \frac{2 \cdot \nu_{21}}{E_1}\right) \cdot \operatorname{sen}^2\theta \cdot \cos^2\theta + \frac{1}{E_2} \cdot \operatorname{sen}^4\theta$$

$$\frac{1}{E_y} = \frac{1}{E_1} \cdot \operatorname{sen}^4\theta + \left(\frac{1}{G_{12}} - \frac{2 \cdot \nu_{21}}{E_1}\right) \cdot \operatorname{sen}^2\theta \cdot \cos^2\theta + \frac{1}{E_2} \cdot \cos^4\theta$$

$$\frac{1}{G_{xy}} = 2 \cdot \left(\frac{2}{E_1} + \frac{2}{E_2} - \frac{1}{G_{12}} + \frac{4 \cdot \nu_{21}}{E_1}\right) \cdot \operatorname{sen}^2\theta \cdot \cos^2\theta + \frac{1}{G_{12}} \cdot \left(\operatorname{sen}^4\theta + \cos^4\theta\right)$$

$$\nu_{yx} = E_x \cdot \left[\frac{\nu_{21}}{E_1} \cdot \left(\operatorname{sen}^4\theta + \cos^4\theta\right) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right) \cdot \operatorname{sen}^2\theta \cdot \cos^2\theta\right]$$

$$\nu_{xy} = E_y \cdot \left[\frac{\nu_{21}}{E_1} \cdot \left(\operatorname{sen}^4\theta + \cos^4\theta\right) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right) \cdot \operatorname{sen}^2\theta \cdot \cos^2\theta\right]$$

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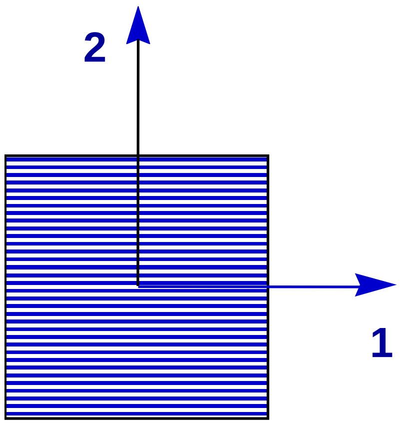
Matriz de flexibilidad de una lámina en ejes locales



$$\{\varepsilon\} = [S] \cdot \{\sigma\}$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{SS} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix}$$

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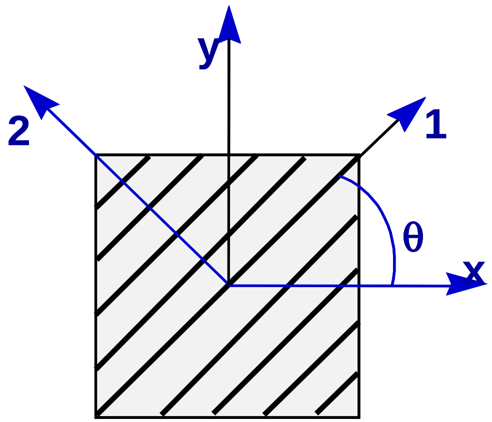
Matrices de flexibilidad

Matriz de flexibilidad de una lámina en ejes globales



$$\{\varepsilon\} = [\bar{S}]\cdot\{\sigma\}$$

$$[S] = \begin{bmatrix} \bar{S}_{xx} & \bar{S}_{xy} & \bar{S}_{xs} \\ \bar{S}_{yx} & \bar{S}_{yy} & \bar{S}_{ys} \\ \bar{S}_{sx} & \bar{S}_{sy} & \bar{S}_{ss} \end{bmatrix}$$

$$\bar{S}_{xx} = S_{xx}\cos^4\theta + (2\cdot S_{xy} + S_{ss})\,\mathrm{sen}^2\theta\cos^2\theta + S_{yy}\,\mathrm{sen}^4\theta$$

$$\bar{S}_{yx} = (S_{xx} + S_{yy} - S_{ss})\,\mathrm{sen}^2\theta\cos^2\theta + S_{xy}(\mathrm{sen}^4\theta + \cos^4\theta)$$

$$\bar{S}_{yy} = S_{xx}\,\mathrm{sen}^4\theta + (2S_{xy} + S_{ss})\,\mathrm{sen}^2\theta\cos^2\theta + S_{yy}\cos^4\theta$$

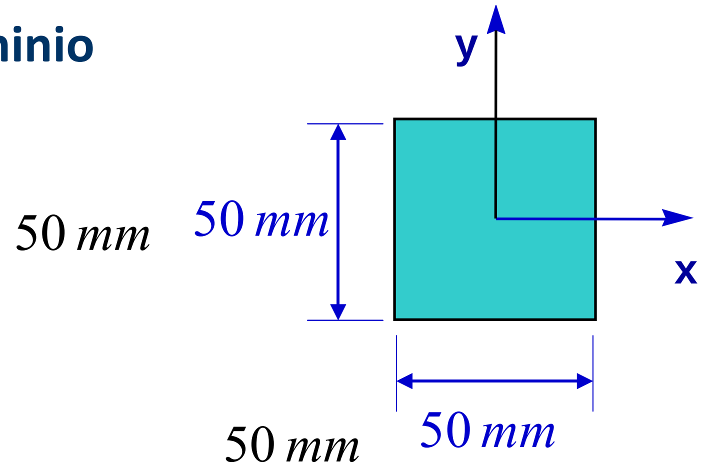
$$\bar{S}_{xs} = (2S_{xx} - 2S_{xy} - S_{ss})\,\mathrm{sen}\theta\cos^3\theta - (2S_{yy} - 2S_{xy} - S_{ss})\,\mathrm{sen}^3\theta\cos\theta$$

$$\bar{S}_{ys} = (2S_{xx} - 2S_{xy} - Q_{ss})\,\mathrm{sen}^3\theta\cos\theta - (-2S_{xy} + 2S_{yy} - 2S_{ss})\,\mathrm{sen}\theta\cos^3\theta$$

$$\bar{S}_{ss} = 2(2S_{xx} + 2S_{yy} - 4S_{xy} - S_{ss})\,\mathrm{sen}^2\theta\cos^2\theta + S_{ss}(\mathrm{sen}^4\theta + \cos^4\theta)$$

Ejemplo

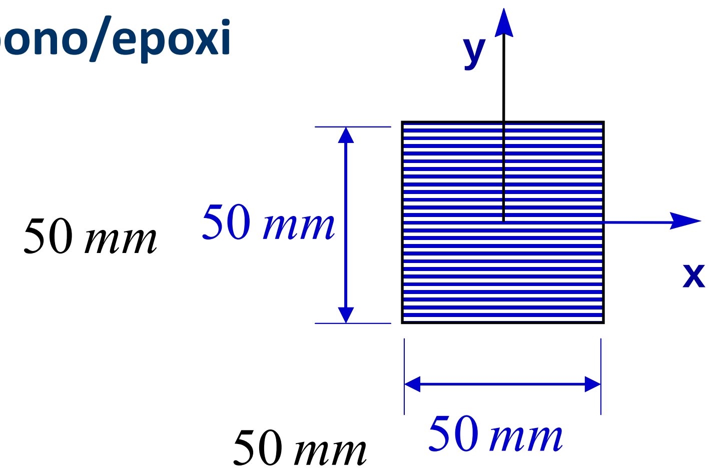
Aluminio



E = 72,4 GPa

ν = 0,3

Carbono/epoxi



E_1 = 155,04 GPa

E_2 = 12,11 GPa

G_{12} = 4,41 GPa

ν_{21} = 0,248

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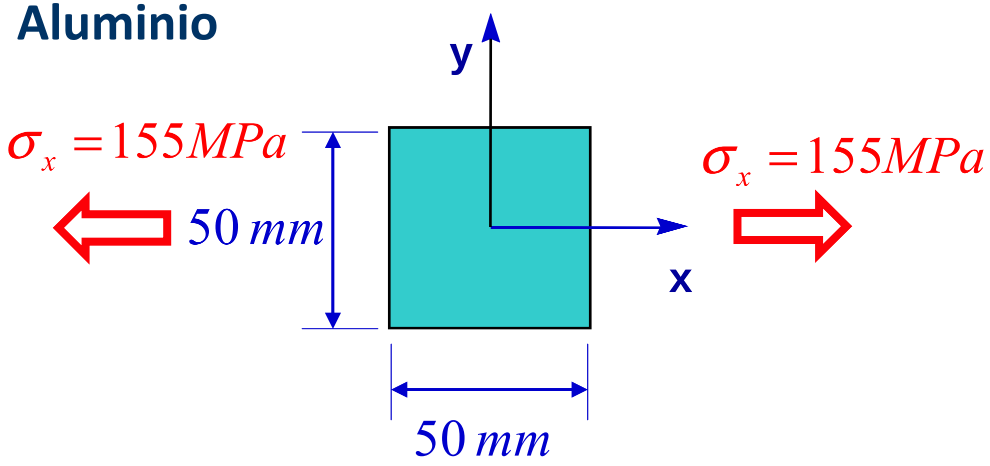
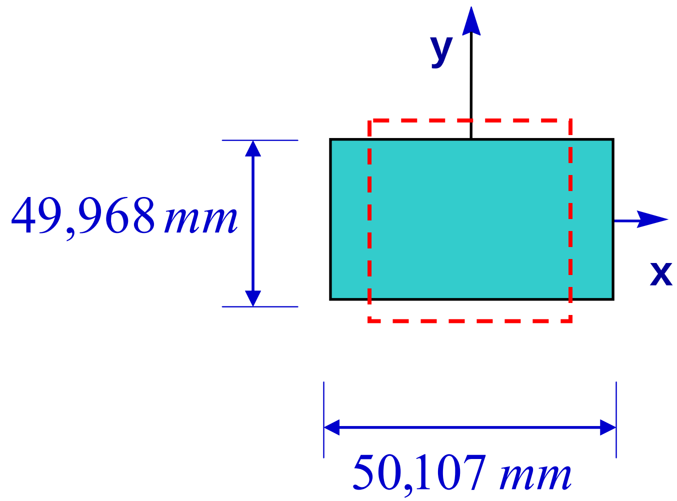
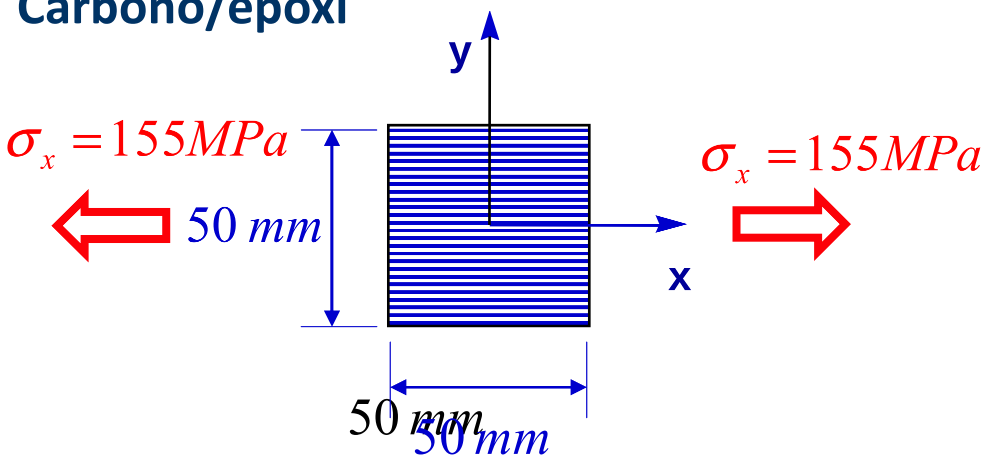
Ejemplo

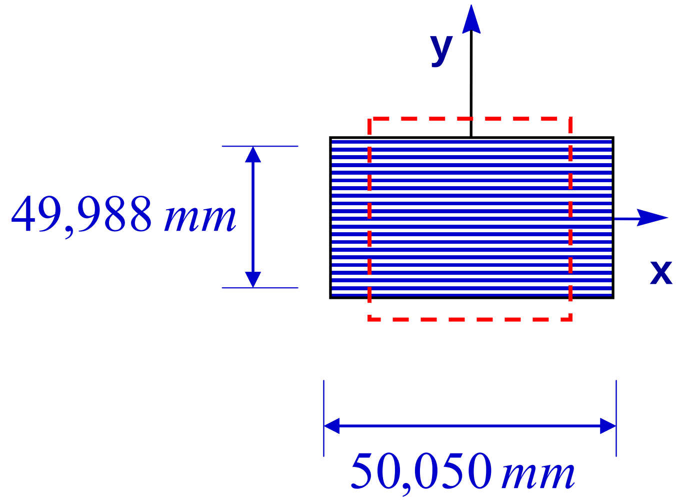
Aluminio





Carbono/epoxi





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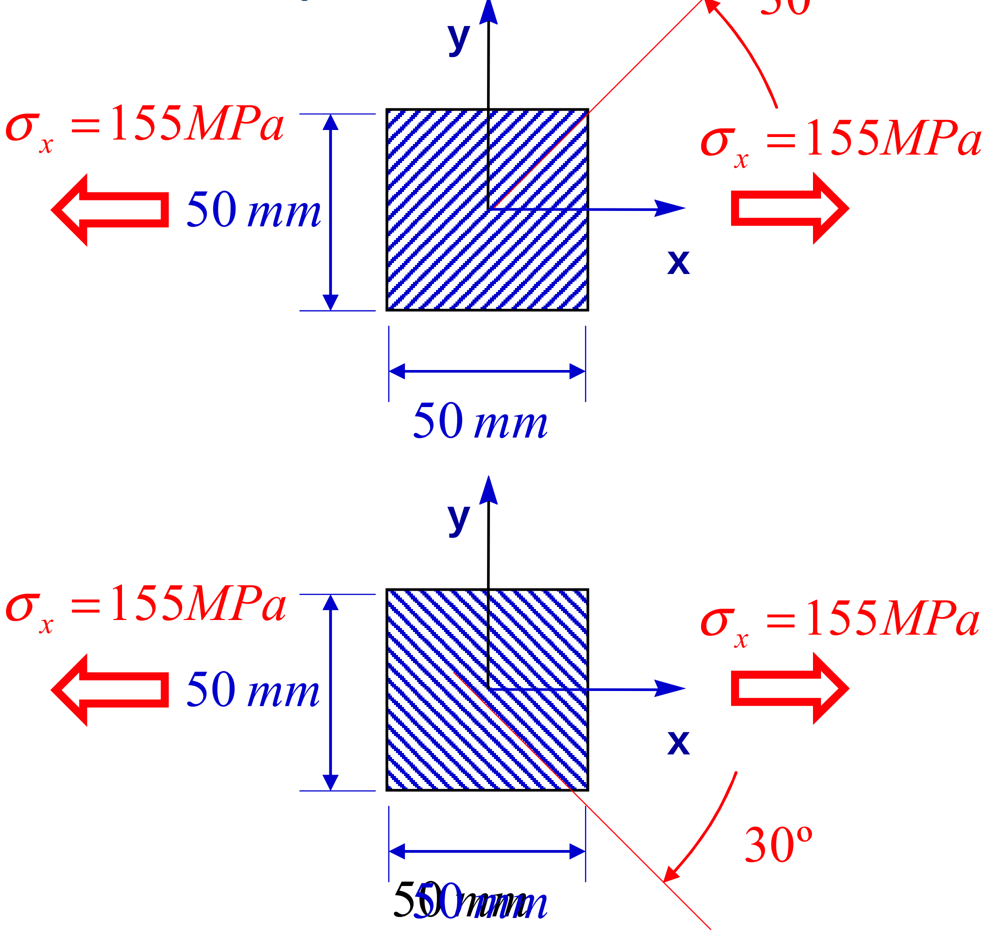
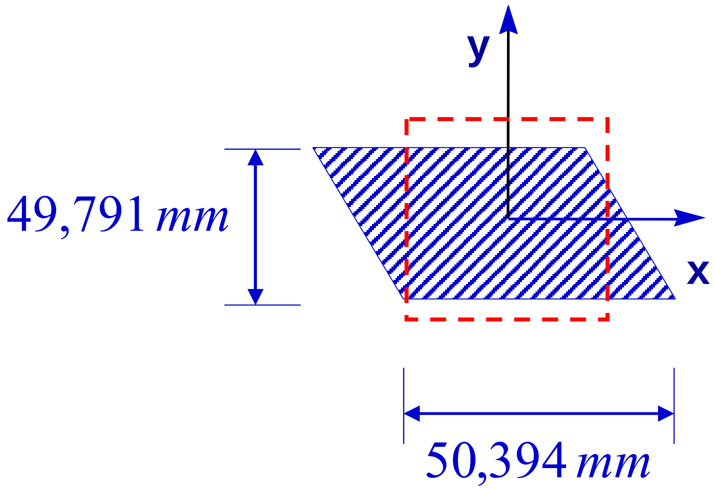
Ejemplo

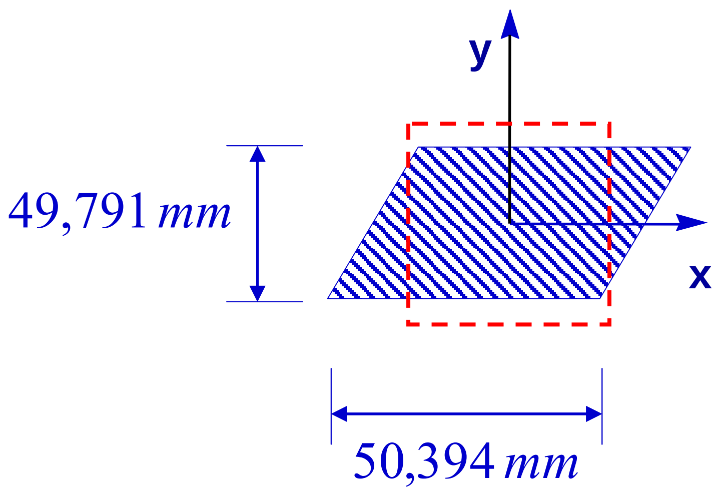
Carbono/epoxi



Incremento de ángulo: 0,553°



Decremento de ángulo: 0,553°



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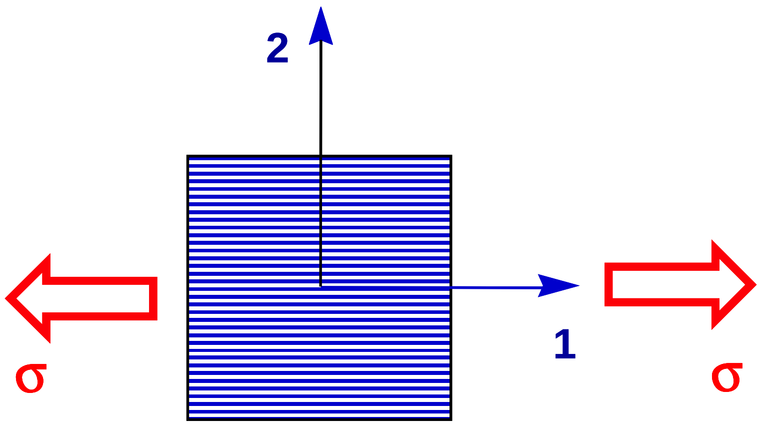
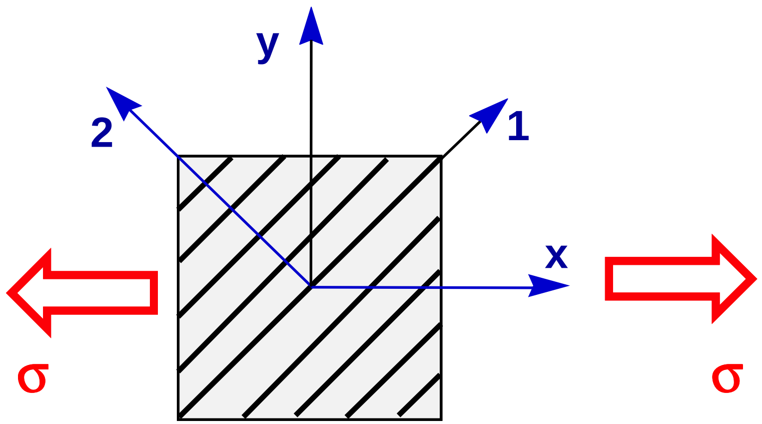
Ejemplo

Cargas de tracción





$$\varepsilon_1 = \frac{\sigma}{E_1}$$

$$\varepsilon_2 = -\frac{\nu_{21}}{E_1} \cdot \sigma$$

$$\{\varepsilon\} = [S] \cdot \{\sigma\}$$

$$\varepsilon_x = \bar{S}_{xx} \cdot \sigma$$

$$\varepsilon_y = \bar{S}_{xy} \cdot \sigma$$

$$\gamma_{xy} = \bar{S}_{xs} \cdot \sigma$$

$$\{\varepsilon\} = [\bar{S}] \cdot \{\sigma\}$$

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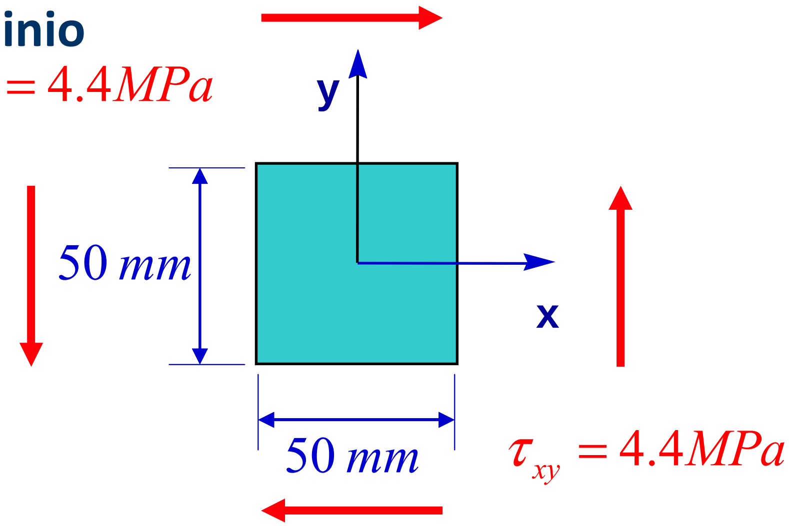
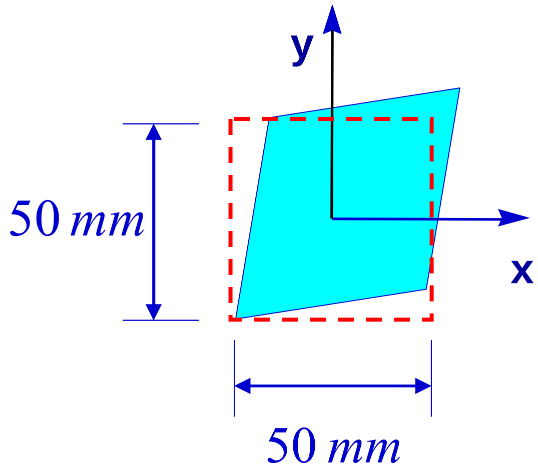
Ejemplo

Aluminio

$\tau_{xy} = 4.4MPa$

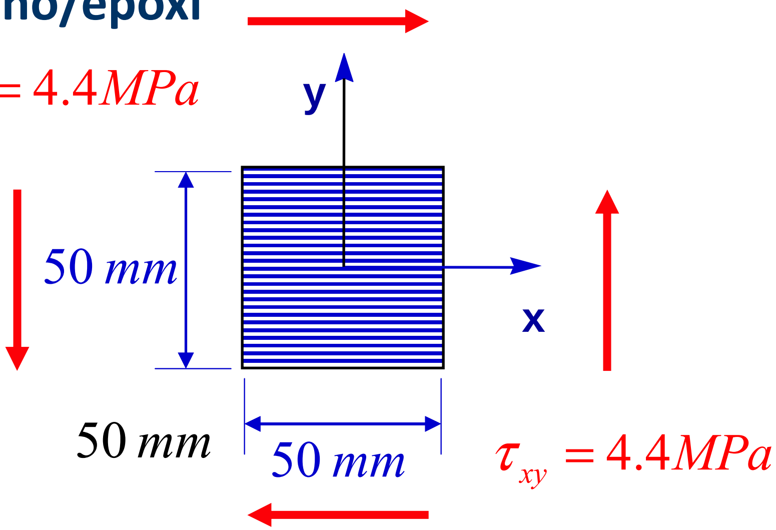


Incremento de ángulo: 0,00905°

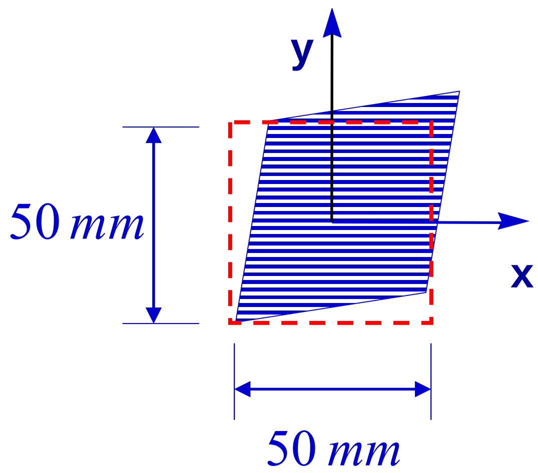


Carbono/epoxi

$\tau_{xy} = 4.4MPa$



Decremento de ángulo: 0,0573°



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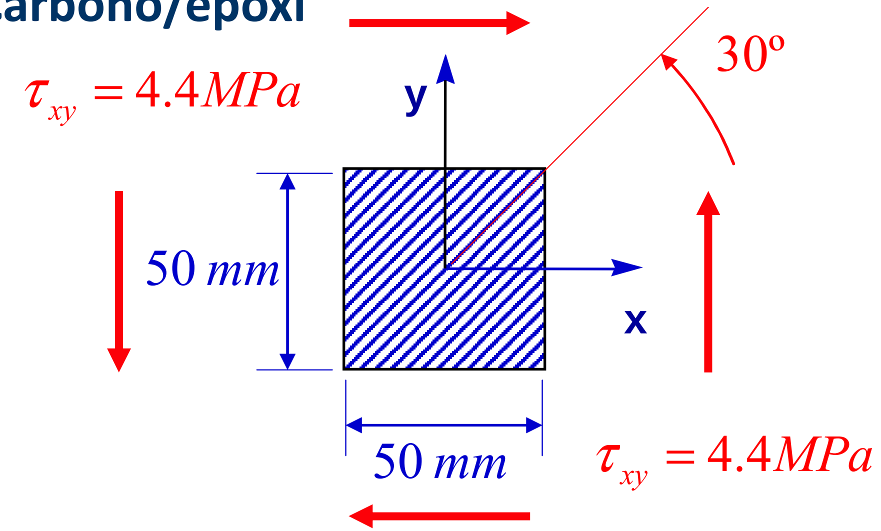
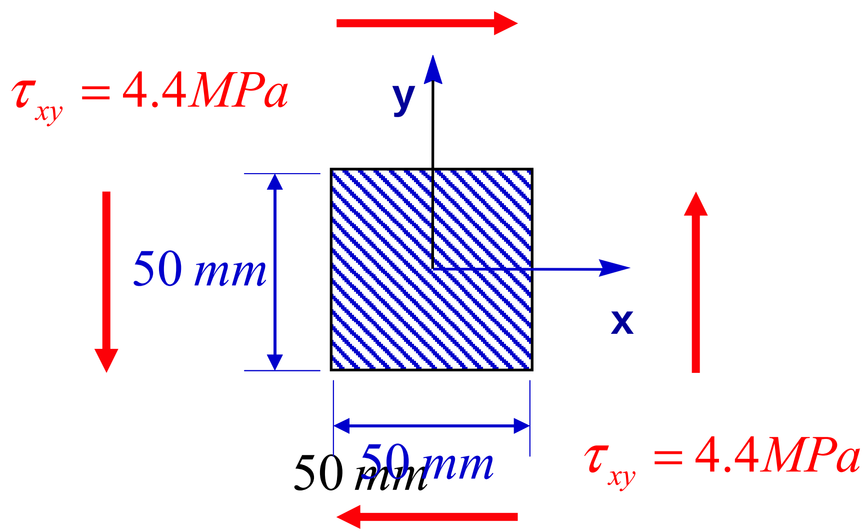
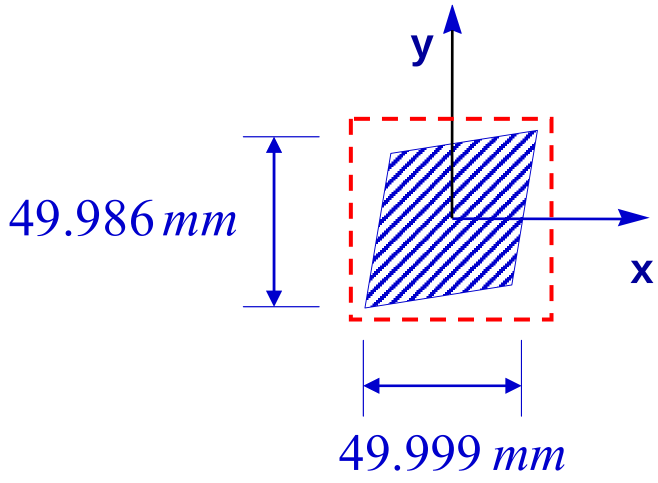
Ejemplo

Carbono/epoxi

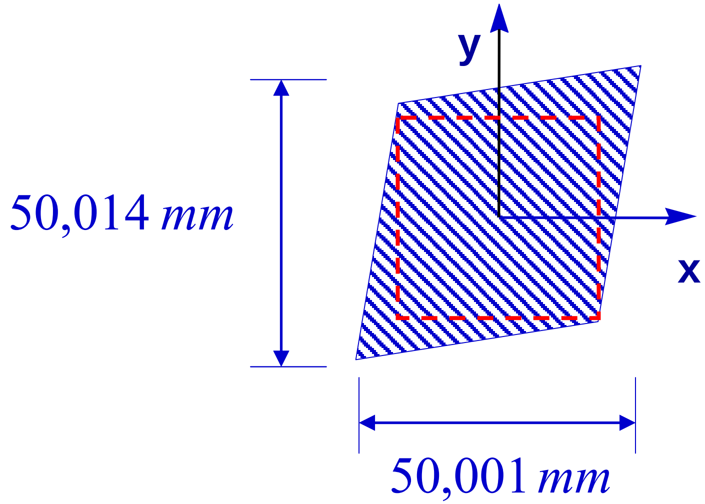




Incremento de ángulo: 0,0318°



Decremento de ángulo: 0,0573°



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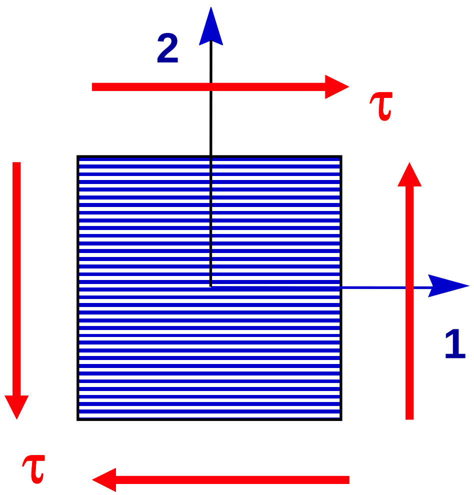
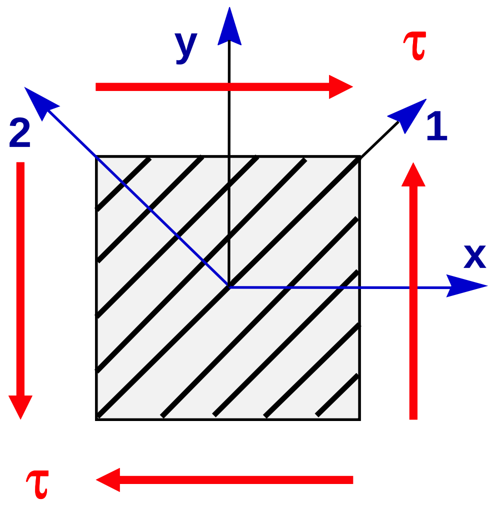
Ejemplo

Cargas de cortadura





$$\gamma_{12} = \frac{\tau}{G_{12}}$$

$$\{\varepsilon\} = [S] \cdot \{\sigma\}$$

$$\varepsilon_x = \bar{S}_{xs} \cdot \tau$$

$$\varepsilon_y = \bar{S}_{xs} \cdot \tau$$

$$\gamma_{xy} = \bar{S}_{ss} \cdot \tau$$

$$\{\varepsilon\} = [\bar{S}] \cdot \{\sigma\}$$

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