

# ANÁLISIS FRECUENCIAL

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## Diagrama de Bode asintótico.

1. Diagrama asintótico de Bode de términos constantes.
  2. Diagrama asintótico de Bode de polos/ceros en el origen.
  3. Diagrama asintótico de Bode de polos/ceros reales negativos.
  4. Diagrama asintótico de Bode de polos/ceros reales positivos.
  5. Diagrama asintótico de Bode de polos/ceros complejos negativos.
  6. Diagrama asintótico de Bode de polos/ceros imaginarios.
  7. Diagrama asintótico de Bode de sistemas complejos.
  8. Interpretación de un Diagrama de Bode.
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# Bibliografía

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- Ogata, K., "Ingeniería de control moderna", Ed. Prentice-Hall.
    - Capítulo 8
  - Dorf, R.C., "Sistemas modernos de control", Ed. Addison-Wesley.
    - Capítulo
  - Kuo, B.C., "Sistemas de control automático", Ed. Prentice Hall.
    - Capítulo 9
  - F. Matía y A. Jiménez, "Teoría de Sistemas", Sección de Publicaciones Universidad Politécnica de Madrid
    - Capítulo 9
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# DIAGRAMA DE BODE

- El diagrama de Bode se puede obtener como suma de los diagramas de Bode de cada uno de los factores de la función de transferencia

$$G(s) = k \frac{\prod (s - z_i)}{\prod (s - p_i)}$$

$$G(j\omega) = k \frac{\prod (j\omega - z_i)}{\prod (j\omega - p_i)} \Rightarrow$$

$$\Rightarrow \begin{cases} |G(j\omega)| = |k| \frac{\prod |j\omega - z_i|}{\prod |j\omega - p_i|} \\ \arg(G(j\omega)) = \arg(k) + \sum \arg(j\omega - z_i) - \sum \arg(j\omega - p_i) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \log |G(j\omega)| = \log |k| + \sum \log |j\omega - z_i| - \sum \log |j\omega - p_i| \\ \arg(G(j\omega)) = \arg(k) + \sum \arg(j\omega - z_i) - \sum \arg(j\omega - p_i) \end{cases}$$

# DIAGRAMA DE BODE

- Una vez factorizada la F.T., el diagrama de Bode total es la suma de los diagramas de Bode sencillos.

$$G(s) = K \cdot s^{\pm N} \cdot \left[ \prod \frac{p_i}{s + p_i} \right] \cdot \left[ \prod \frac{s + c_i}{c_i} \right] \cdot \left[ \prod \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \right]$$

Diagram illustrating the factorization of the transfer function  $G(s)$  into its constituent parts:

- Término constante**: Points to the gain  $K$ .
- Polos/ceros En el origen**: Points to the term  $s^{\pm N}$ .
- polos reales**: Points to the product term  $\prod \frac{p_i}{s + p_i}$ .
- ceros reales**: Points to the product term  $\prod \frac{s + c_i}{c_i}$ .
- Pares de polos complejos conjugados**: Points to the product term  $\prod \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ .

# TÉRMINOS CONSTANTES

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 $k$ 

$$k = \begin{cases} || = |k| = 20 \log |k| \text{ dB} \\ \varphi = \begin{cases} 0^{\circ} & k > 0 \\ 180^{\circ} & k < 0 \end{cases} \end{cases}$$

# TÉRMINOS CONSTANTES

- Ganancia

$$20 \log |k| \text{ dB}$$



$\omega (\log)$

*Ej* :  $G (s) = 10 \rightarrow G (j\omega) = 10 \Rightarrow |G (j\omega)| = |10| = 10$

$20 \log |G (j\omega)| = 20 \log (10) = 20 \text{ dB}$

- Desfase

$$180^\circ (k < 0)$$



$$0^\circ (k > 0)$$



$\omega (\log)$

# POLOS/CEROS EN EL ORIGEN

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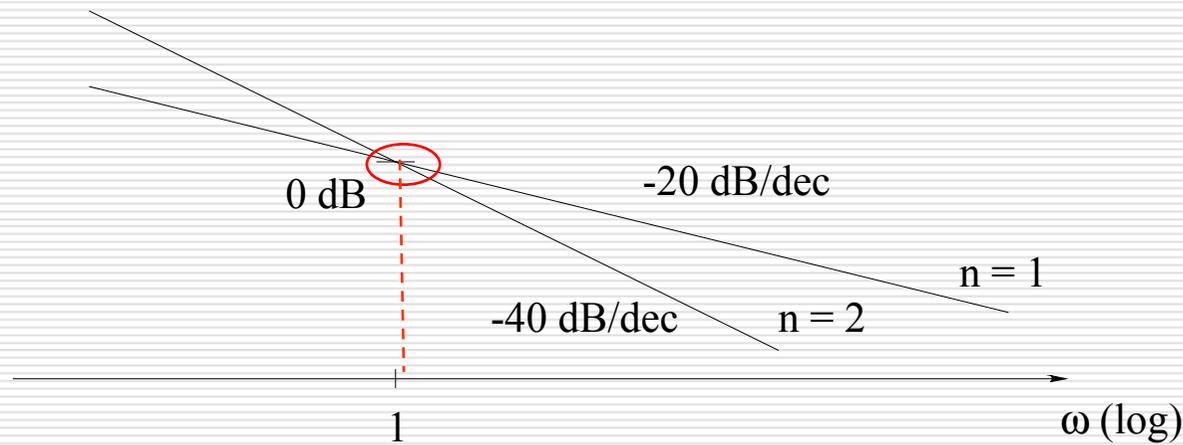
$$\frac{1}{s^n}$$

$$\frac{1}{(j\omega)^n} = \begin{cases} || = \frac{1}{\omega^n} = -20n \log(\omega) \text{ dB} \\ \varphi = -n90^\circ \end{cases}$$

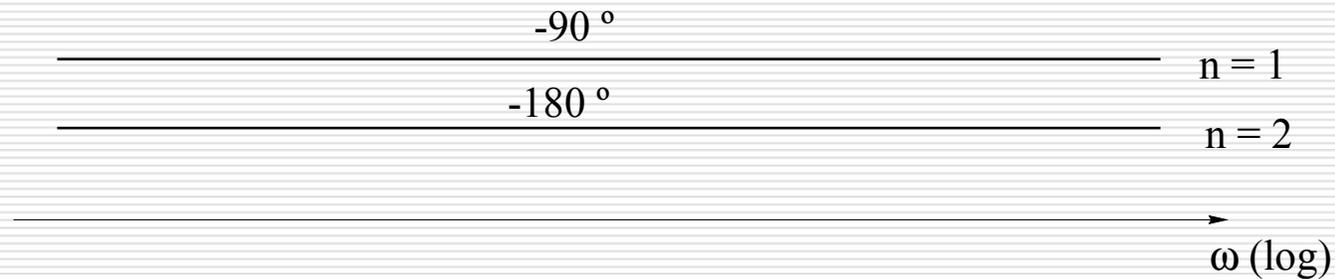
$$\frac{1}{(j\omega)^n} = \begin{cases} 20 \log \left| \frac{1}{(j\omega)^n} \right| = 20 \log \frac{1}{\omega^n} = 20 \log \omega^{-n} = -20n \log(\omega) \text{ dB} \\ \varphi = -n90^\circ \end{cases}$$

# POLOS/CEROS EN EL ORIGEN

- Ganancia



- Desfase



# POLOS/CEROS REALES NEGATIVOS

$$\frac{1}{(1 + T s)^n}$$

$$\frac{1}{(1 + Tj\omega)^n} = \begin{cases} \omega \rightarrow 0 & 1 = \begin{cases} || = 0 \text{ dB} \\ \varphi = 0^\circ \end{cases} \\ \omega \rightarrow \infty & \frac{1}{(Tj\omega)^n} = \begin{cases} || = \frac{1}{(T\omega)^n} = -20n \log(T\omega) \text{ dB} \\ \varphi = -n90^\circ \end{cases} \end{cases}$$

$$(1 + Tj\omega)^n = \begin{cases} \omega \rightarrow 0 & 1 = \begin{cases} || = 0 \text{ dB} \\ \varphi = 0^\circ \end{cases} \\ \omega \rightarrow \infty & (Tj\omega)^n = \begin{cases} || = (T\omega)^n = 20n \log(T\omega) \text{ dB} \\ \varphi = +n90^\circ \end{cases} \end{cases}$$

# POLOS/CEROS REALES NEGATIVOS

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- Frecuencia de corte

Corte de las dos asíntotas

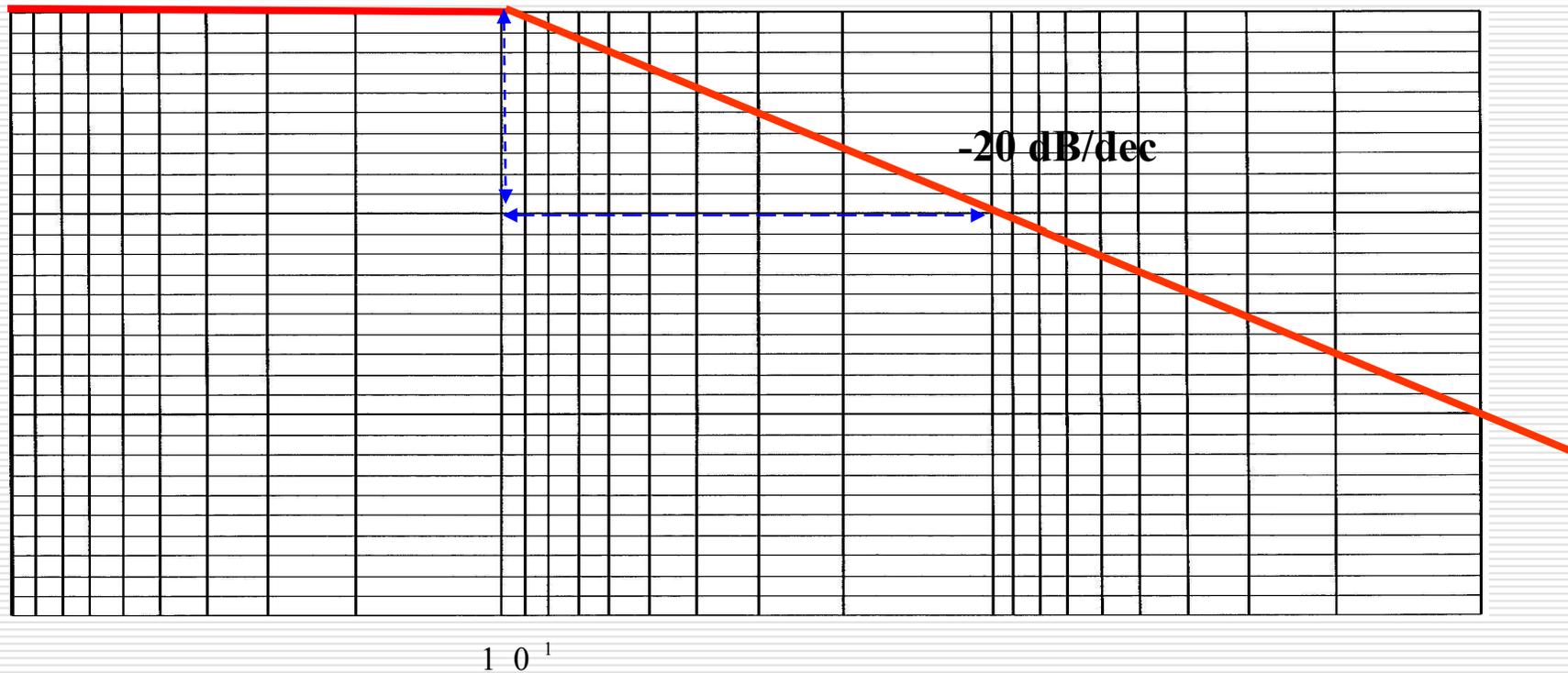
$$0 \text{ db} = -20 \log(\omega_c T)$$

$$\log(\omega_c T) = 0 \Rightarrow \omega_c T = 1$$

$$\omega_c = \frac{1}{T}$$

# POLOS/CEROS REALES NEGATIVOS

$$G(s) = \frac{1}{1 + Ts}, \quad T = 0.1 \quad \omega_c = \frac{1}{T} = \frac{1}{0.1} = 10 \text{ rad/s}$$



# POLOS/CEROS REALES NEGATIVOS

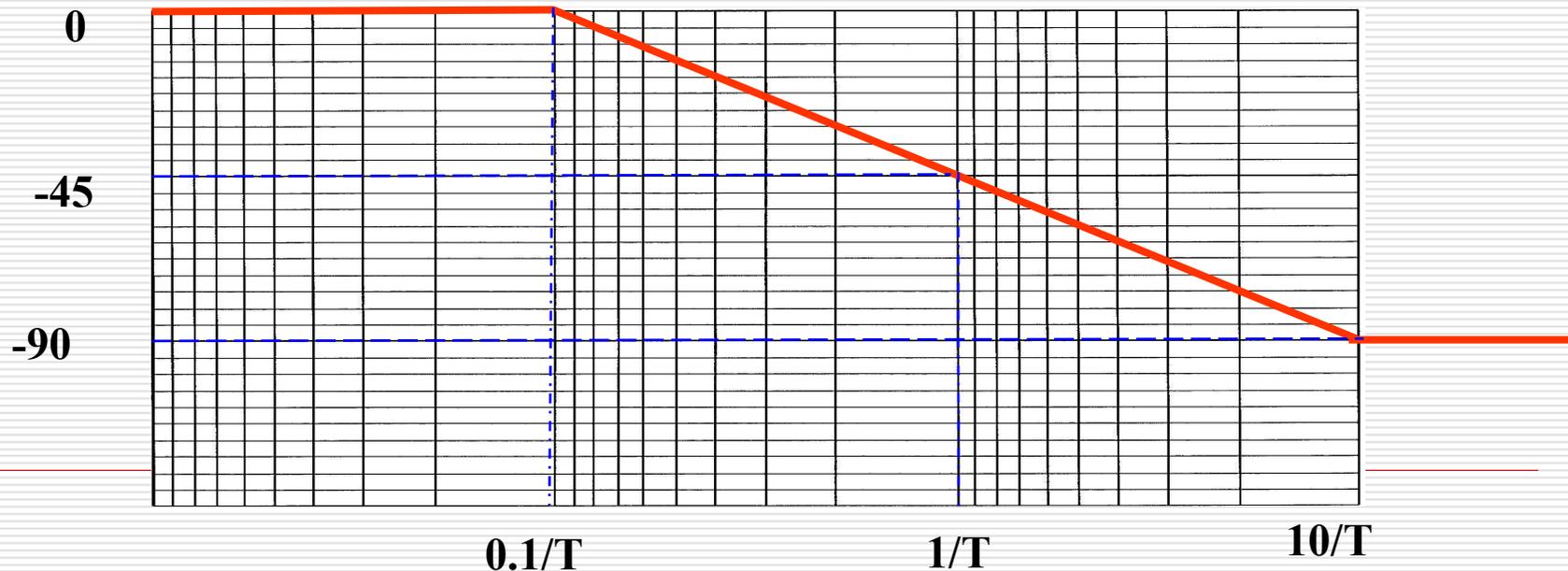
- Diagrama de Fase

$$\varphi = 0 - \arctg(1 + j\omega T) = -\arctg(\omega T)$$

$$\omega = \omega_c = \frac{1}{T} \Rightarrow \varphi = -\arctg\left(\frac{1}{T} \cdot T\right) = -\arctg(1) = -45^\circ$$

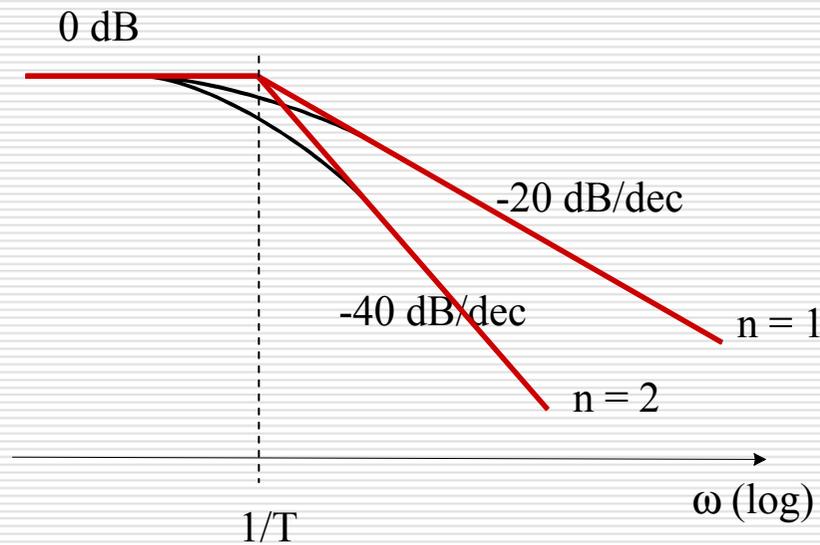
$$\omega = 10^{-1} \omega_c = \frac{10^{-1}}{T} \Rightarrow \varphi = -\arctg\left(\frac{10^{-1}}{T} \cdot T\right) = -\arctg(0.1) = -6^\circ \approx 0^\circ$$

$$\omega = 10^1 \omega_c = \frac{10^1}{T} \Rightarrow \varphi = -\arctg\left(\frac{10^1}{T} \cdot T\right) = -\arctg(10) = -84^\circ \approx 90^\circ$$

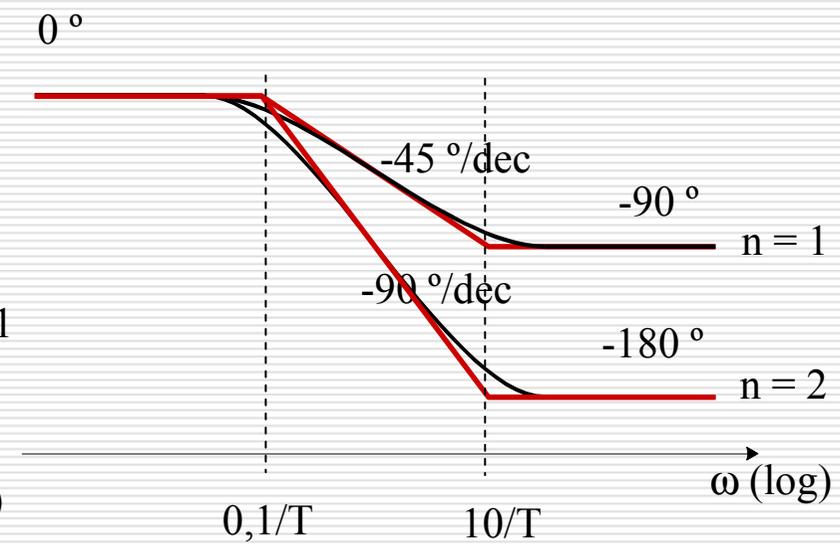


# POLOS/CEROS REALES NEGATIVOS

## ■ Ganancia



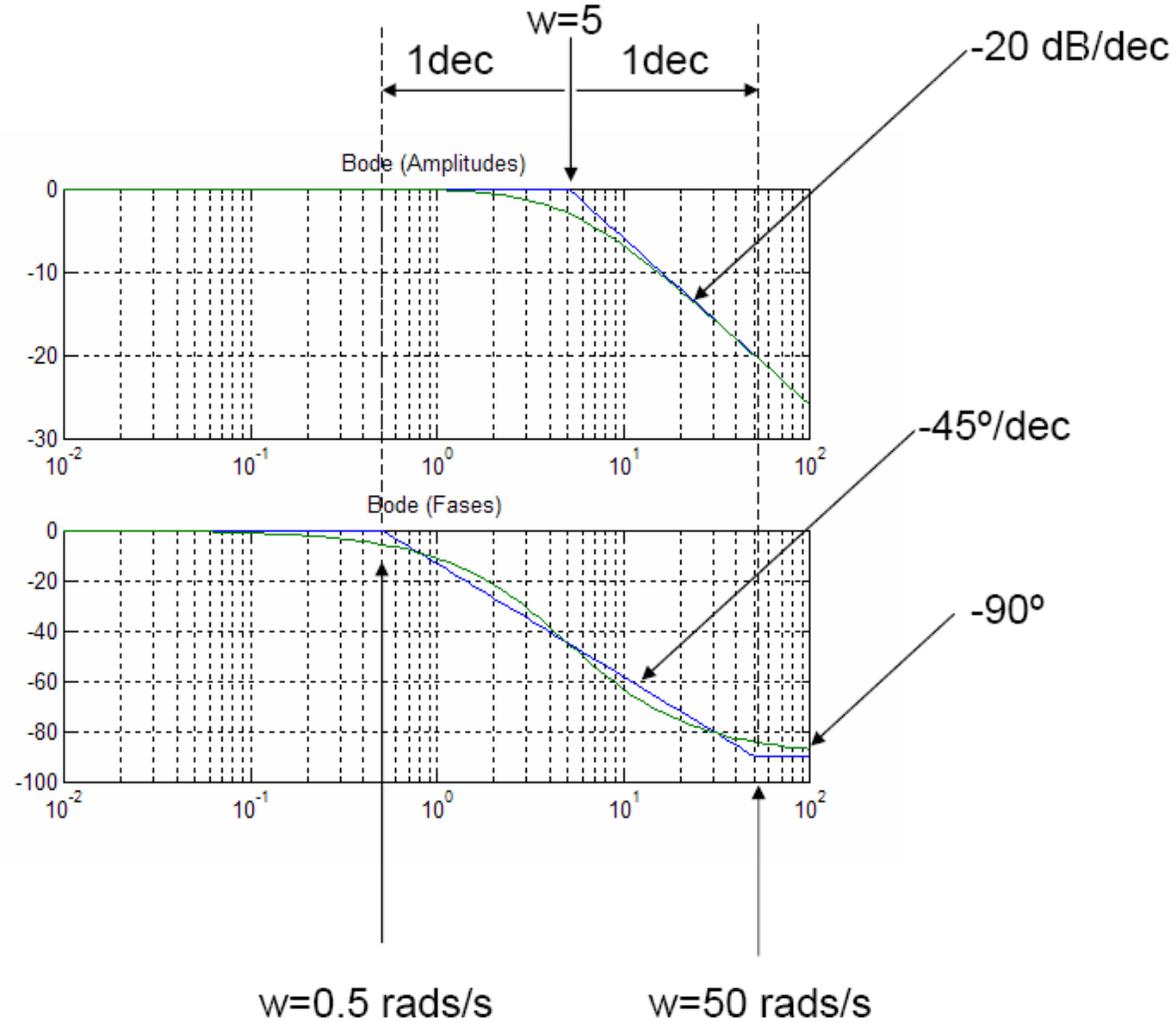
## ■ Desfase



# Diagrama de Bode de polos reales negativos

## Polo real

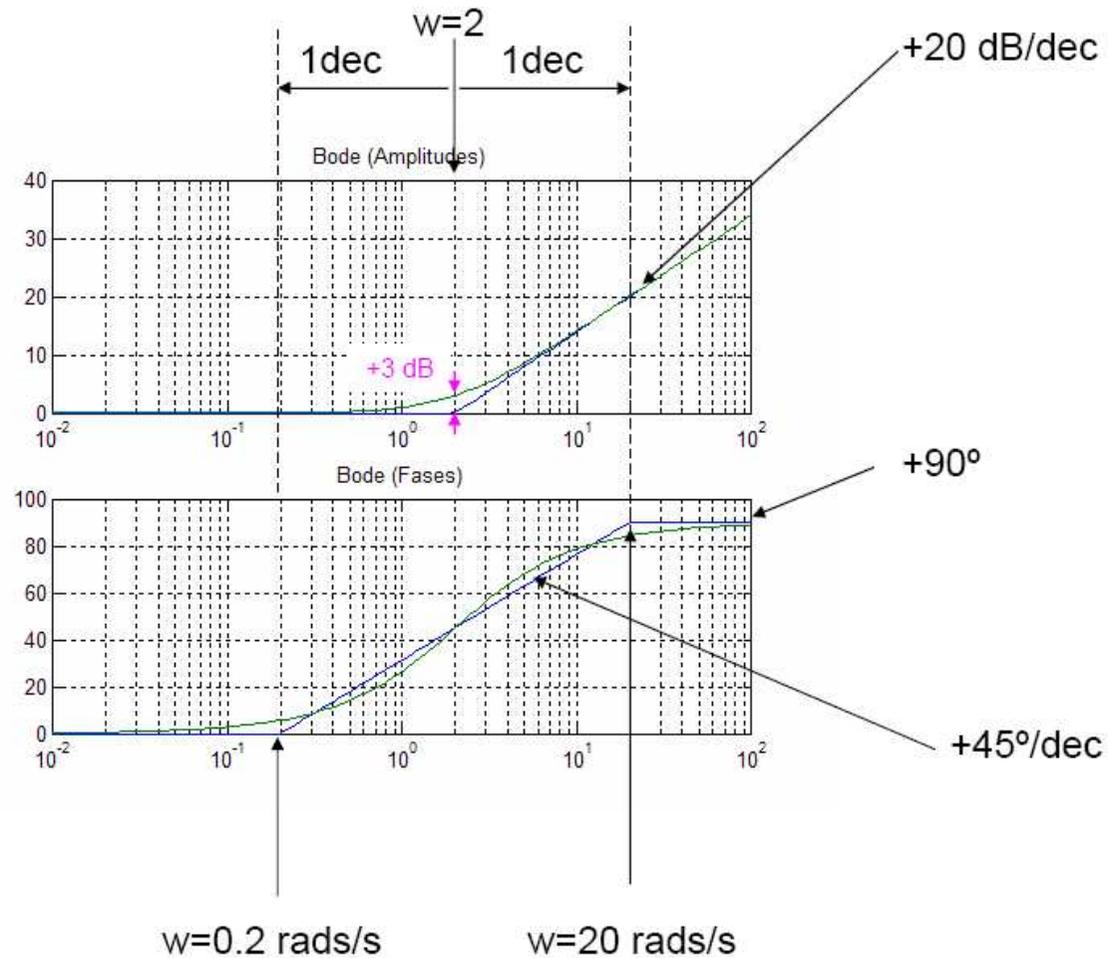
$$G(s) = \frac{5}{s + 5}$$



# Diagrama de Bode de ceros reales negativos

## Cero real

$$G(s) = \frac{s + 2}{2}$$

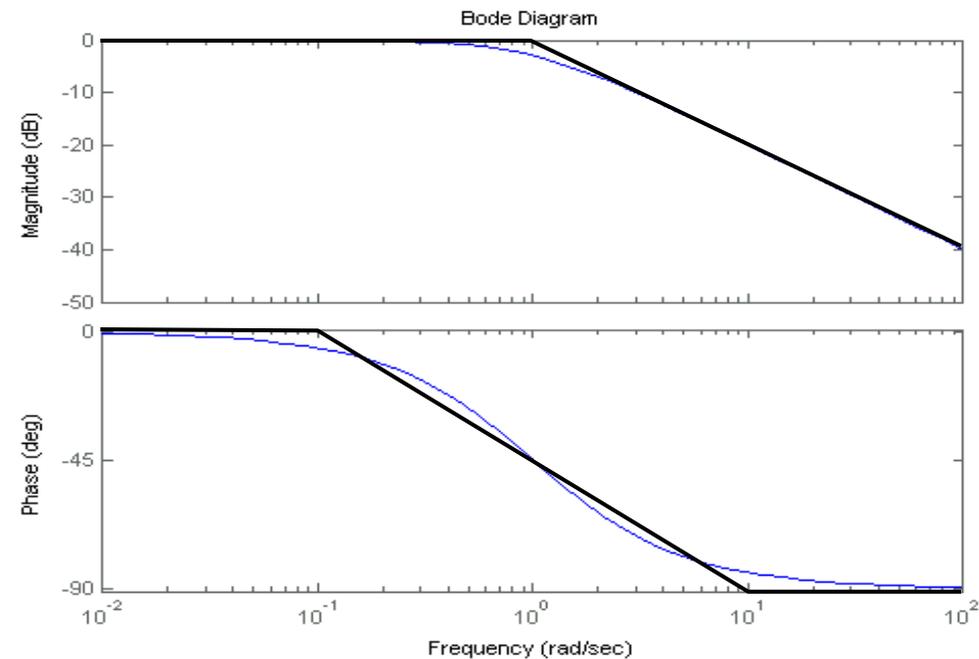


# POLOS/CEROS REALES

$$\frac{1}{s + 1}$$

$$\omega_c = \frac{1}{T}$$

$$\omega_c = \frac{1}{1} = 1 \text{ rad/s} = 10^0 \text{ rad/s}$$

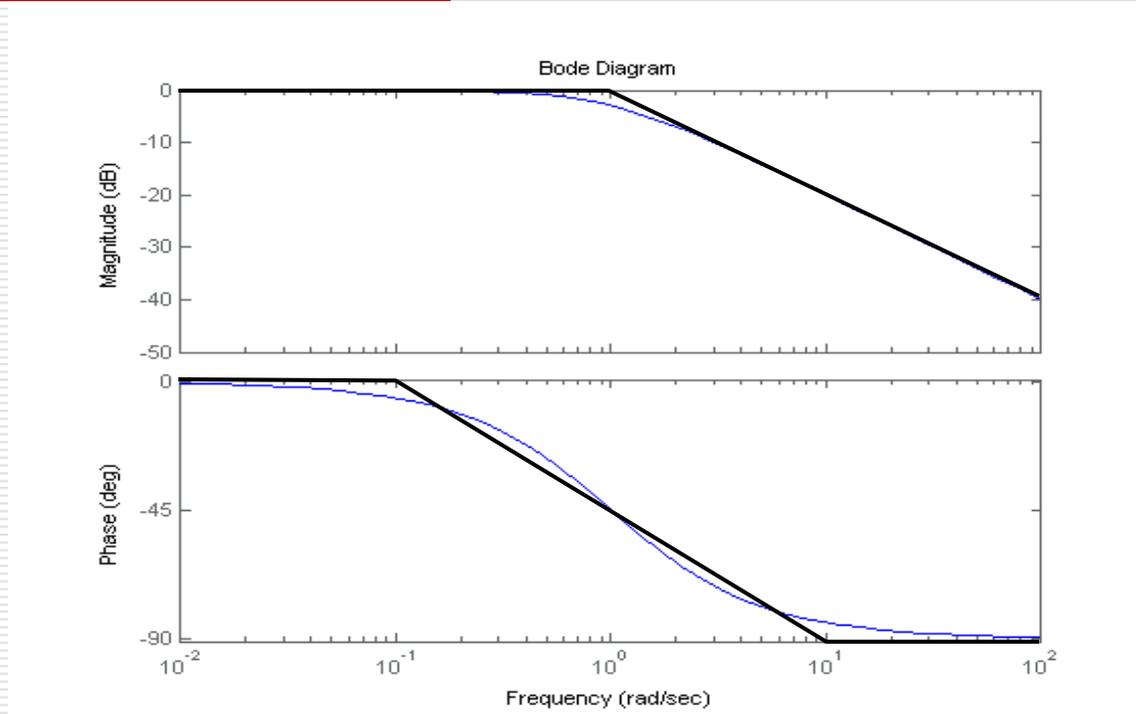


Para la fase:

- Empieza en 0
- Comienza a caer una década antes de  $\omega_c \Rightarrow \omega = 10^{-1} \text{ rad}$
- En  $\omega = \omega_c$ , ha caído  $-45^\circ$
- Termina la caída ( en  $-90^\circ$  ) una década después de  $\omega_c \Rightarrow \omega = 10^1 \text{ rad}$

# Diagrama de Bode Polos/ceros reales

$$\frac{1}{S + 1}$$



$$20 \cdot \log \left| \frac{1}{1 + \omega T} \right| = 20 \log \left( \frac{1}{\sqrt{1 + \omega^2 T^2}} \right) = -20 \log \left( \sqrt{1 + \omega^2 T^2} \right)$$

$$\omega_c = \frac{1}{T}, \quad -20 \log \left( \sqrt{1 + \omega_c^2 T^2} \right) = -20 \log \sqrt{2} \text{ (dB)} \approx -3 \text{ dB}$$

# POLOS/CEROS COMPLEJOS CON PARTE REAL NEGATIVA

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(1 + \frac{2\zeta}{\omega_n}s + \left(\frac{s}{\omega_n}\right)^2\right)^n}$$

$$\frac{1}{\left(1 + \frac{2\zeta}{\omega_n}j\omega + \left(\frac{j\omega}{\omega_n}\right)^2\right)^n} =$$

$$= \begin{cases} \omega \rightarrow 0 & 1 = \begin{cases} || = 0 \text{ dB} \\ \varphi = 0^0 \end{cases} \\ \omega \rightarrow \infty & \left(\frac{j\omega}{\omega_n}\right)^{-2n} = \begin{cases} || = \left(\frac{\omega}{\omega_n}\right)^{-2n} = -40n \log\left(\frac{\omega}{\omega_n}\right) \text{ dB} \\ \varphi = -2n90^0 = -n180^0 \end{cases} \end{cases}$$

# POLOS/CEROS COMPLEJOS CON PARTE REAL NEGATIVA

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- Frecuencia de corte  
Corte de las dos asíntotas

$$0 \text{ db} = -40n \log\left(\frac{\omega}{\omega_n}\right) \text{ dB}$$

$$\log\left(\frac{\omega}{\omega_n}\right) = 0 \Rightarrow \frac{\omega}{\omega_n} = 1$$

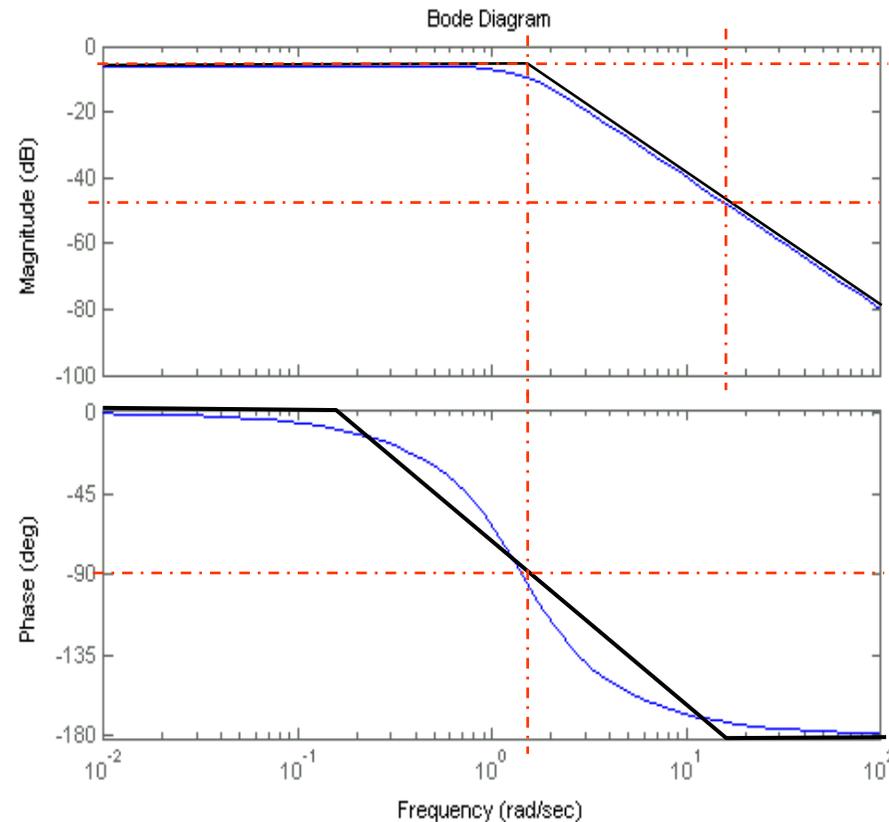
$$\omega_c = \omega_n$$

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# POLOS/CEROS COMPLEJOS CON PARTE REAL NEGATIVA

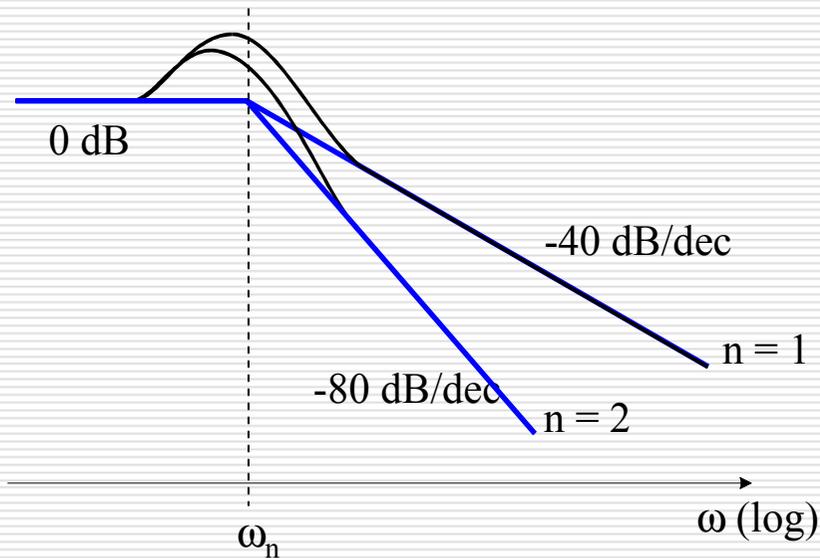
$$\frac{2}{s^2 + 2s + 2}$$

$$\omega_n = \sqrt{2} \quad (\zeta = 0.707)$$

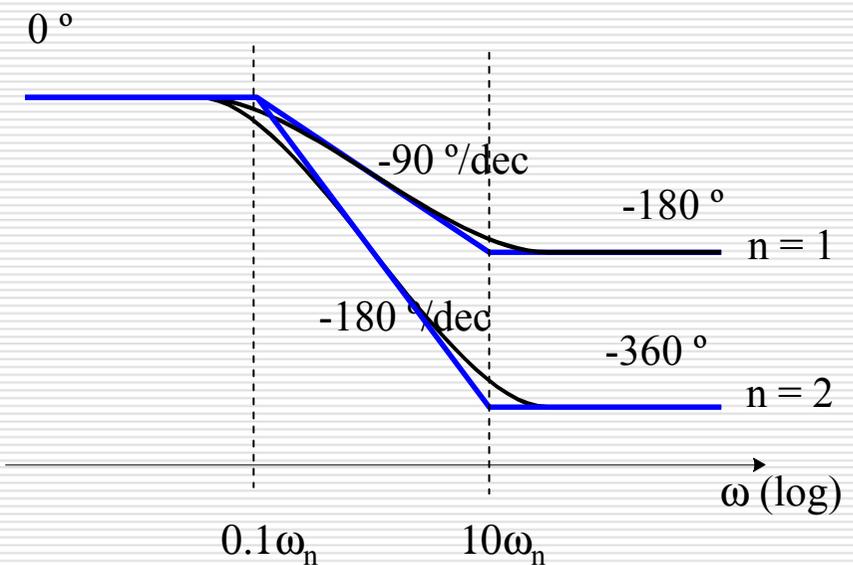


# POLOS COMPLEJOS CON PARTE REAL NEGATIVA

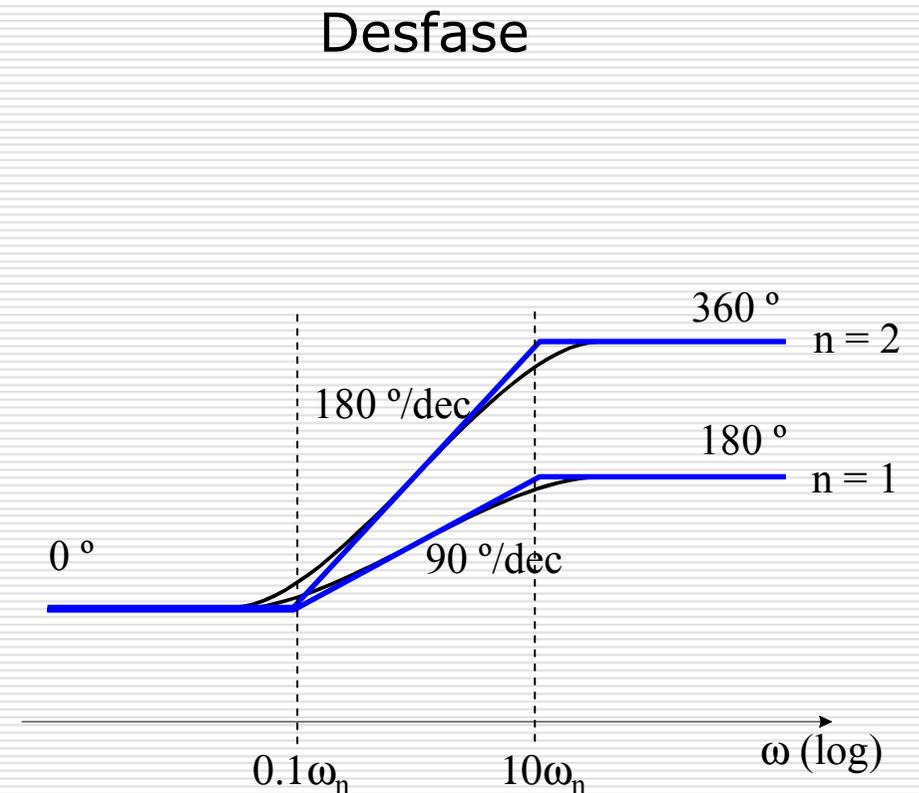
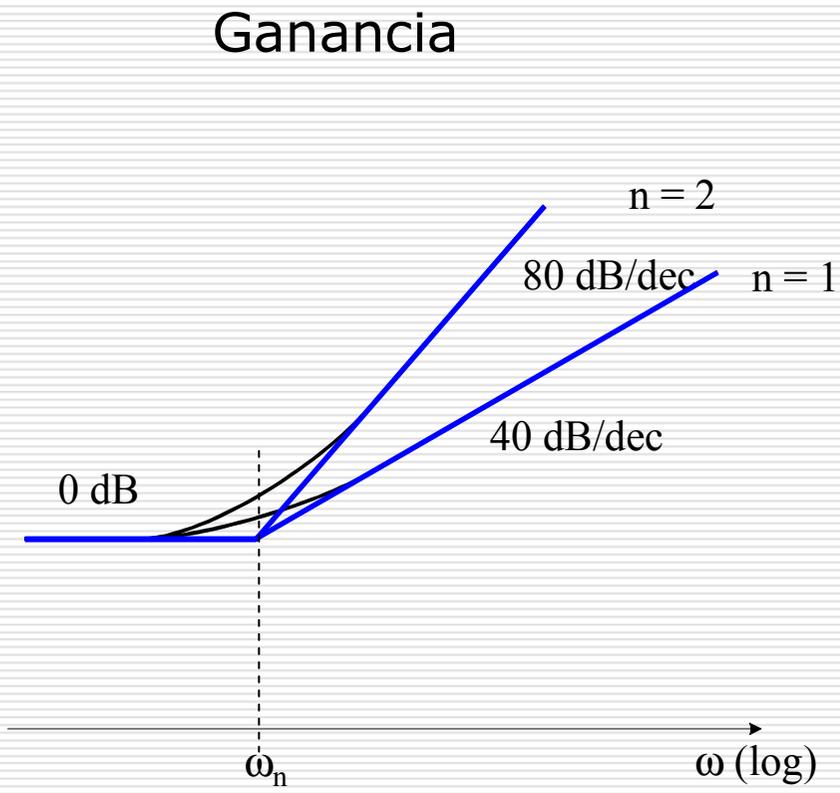
■ **Ganancia**



■ **Desfase**



# CEROS COMPLEJOS CON PARTE REAL NEGATIVA



# POLOS/CEROS COMPLEJOS CON PARTE REAL NEGATIVA

- Frecuencia de resonancia  $\omega_r$

$$\omega_r = \omega_n \sqrt{1 - 2\xi^2} \quad (0 < \xi < 0.707)$$

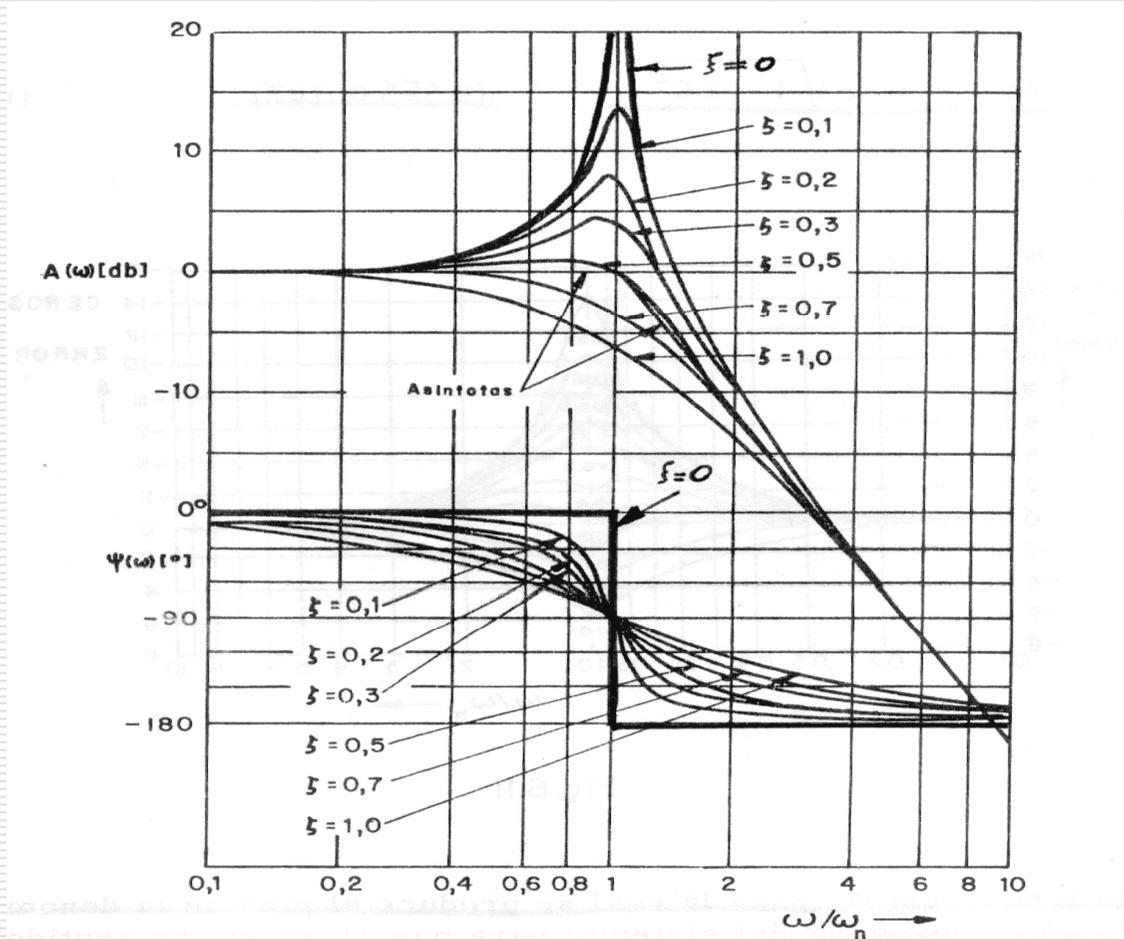
- $\omega_r$  existe cuando:

$$1 - 2\xi^2 > 0 \quad \Rightarrow \quad \xi < \frac{1}{\sqrt{2}} = 0.707$$

- Pico de resonancia en función de  $\xi$

$$|G(j\omega)|_{\max} = |G(j\omega_r)| = M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad (0 < \xi < 0.707)$$

# POLOS/CEROS COMPLEJOS CON PARTE REAL NEGATIVA

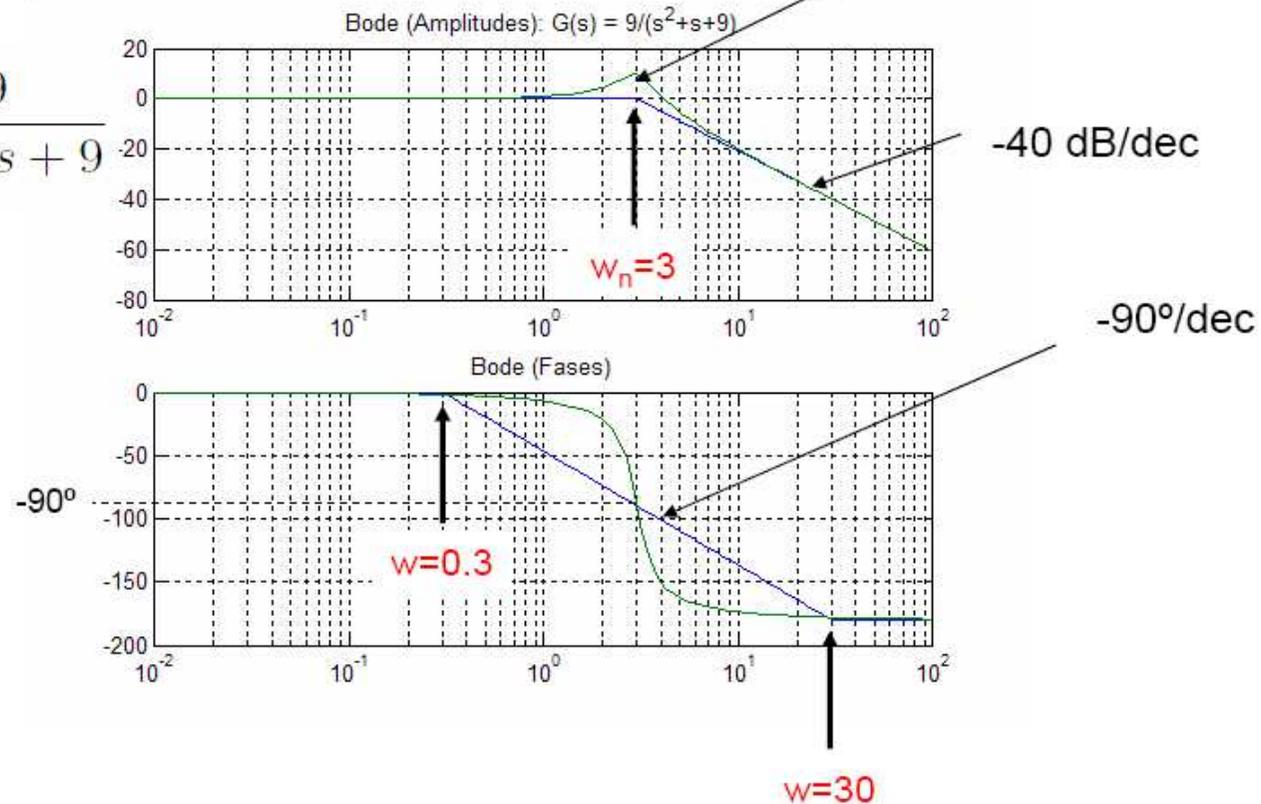


# Diagrama de Bode de polos/ceros positivos

## Polos complejos conjugados

La resonancia depende del factor de amortiguamiento  $\xi$  pequeño  $\rightarrow$  resonancia grande (ver tablas graficas Puente)

$$G(s) = \frac{9}{s^2 + s + 9}$$

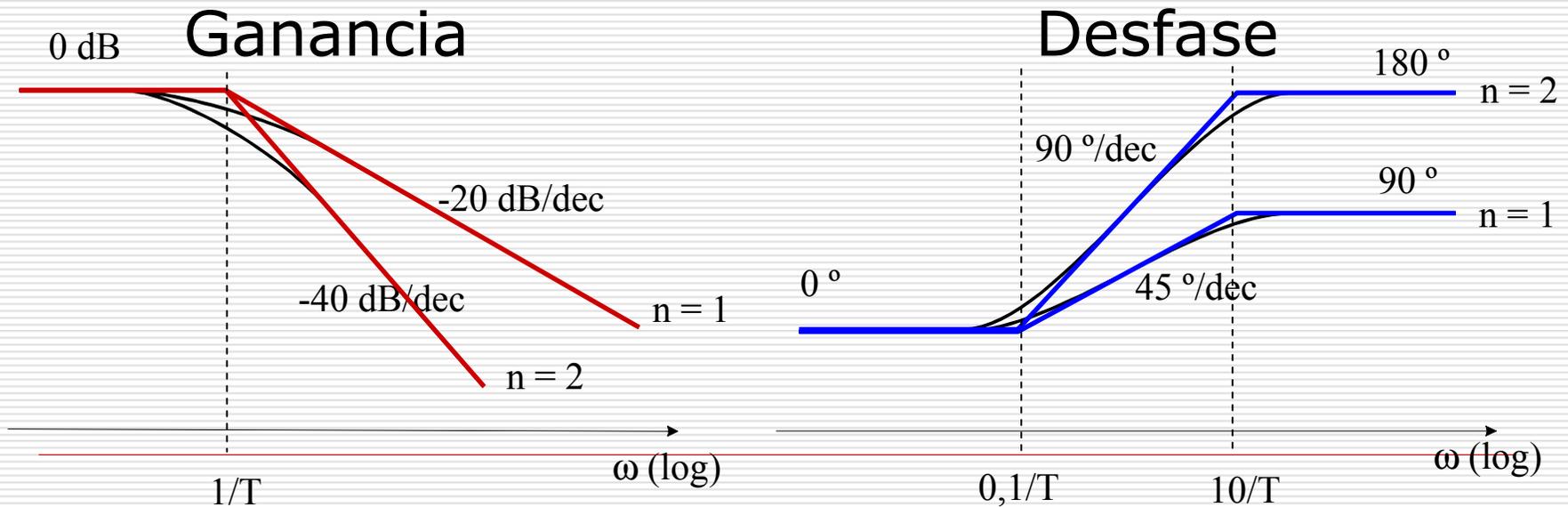


# Diagrama de Bode de polos/ceros positivos

- En general el diagrama en amplitudes es el mismo que con signo negativo y la pendiente en fase es la misma pero de signo contrario.

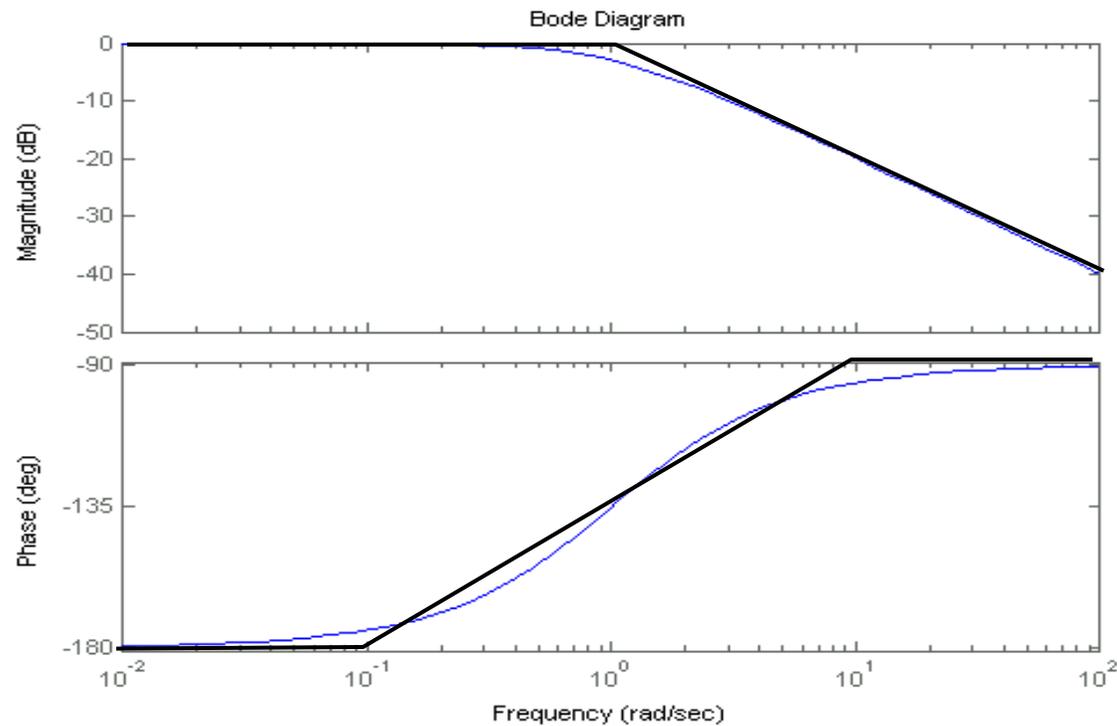
$$\boxed{\frac{1}{(1-Ts)^n}}$$

$$\frac{1}{(1-Tj\omega)^n} = \begin{cases} \omega \rightarrow 0 & 1 = \begin{cases} \| = 0dB \\ \varphi = 0^\circ \end{cases} \\ \omega \rightarrow \infty & \frac{1}{(-Tj\omega)^n} = \begin{cases} \| = \frac{1}{(T\omega)^n} = -20n \log(T\omega)dB \\ \varphi = n90^\circ \end{cases} \end{cases}$$



# POLOS/CEROS REALES POSITIVOS

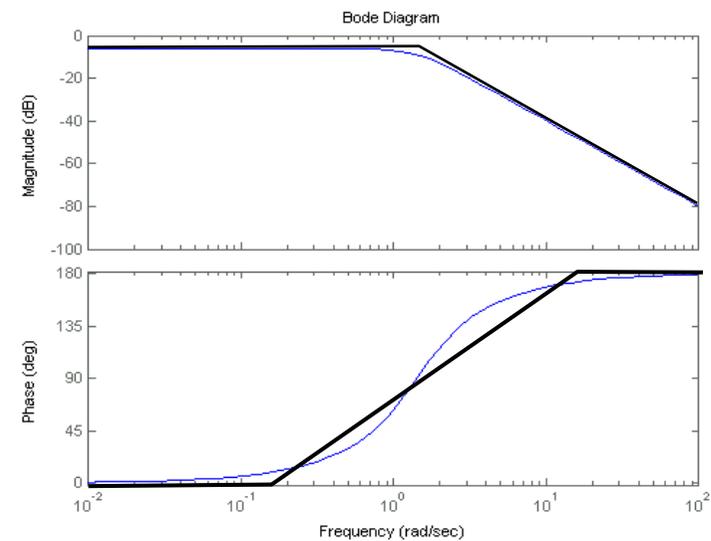
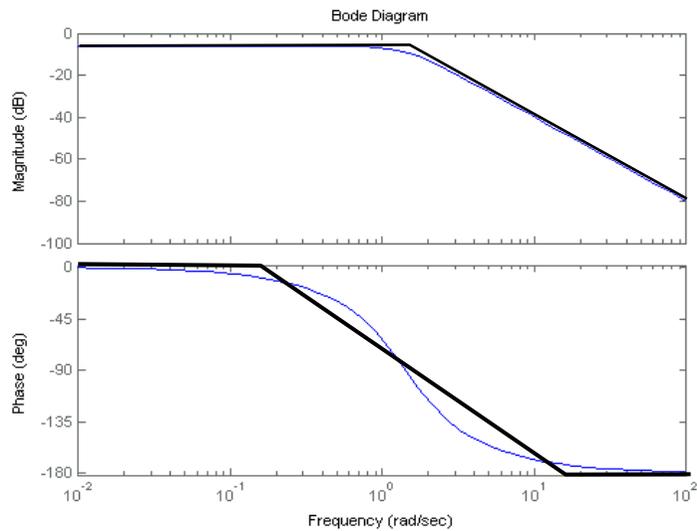
$$G(s) = \frac{1}{s - 1}$$



# POLOS/CEROS COMPLEJOS CON PARTE REAL POSITIVA

$$\frac{1}{S^2 + 2S + 2}$$

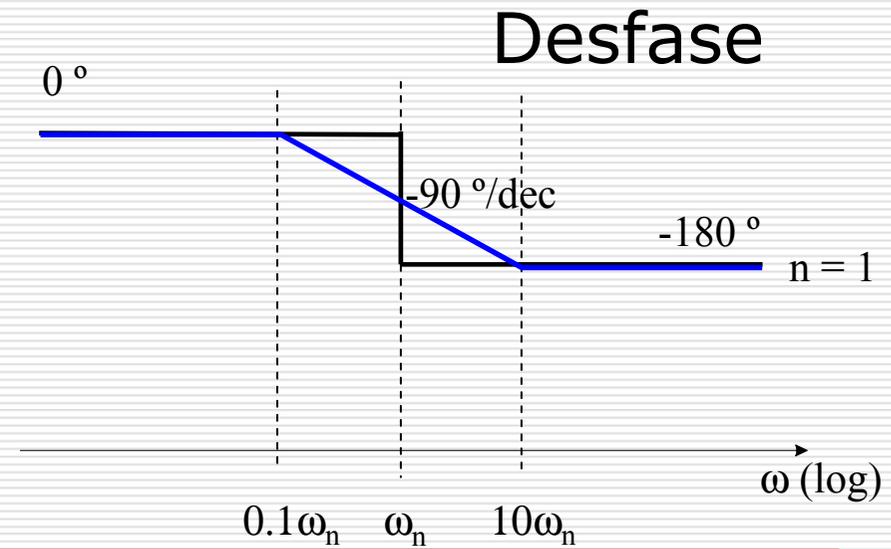
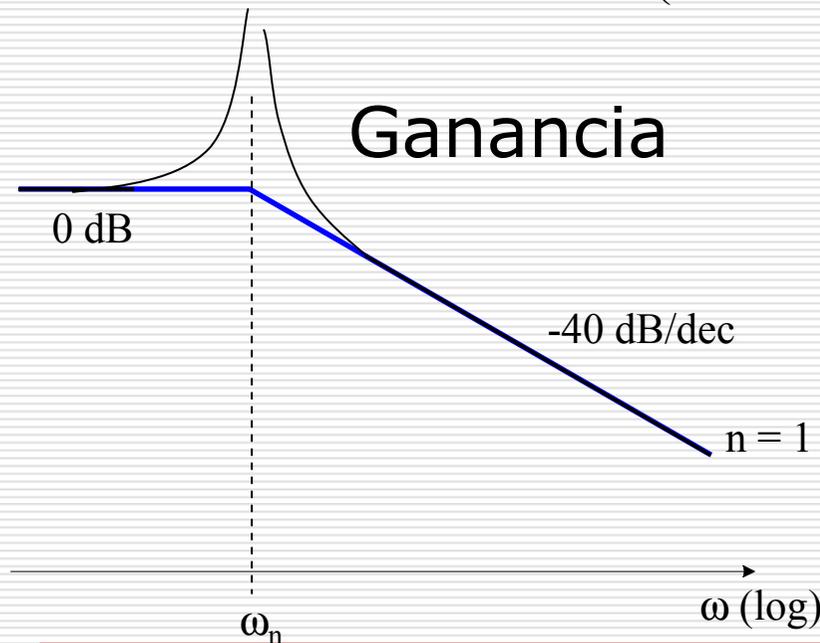
$$\frac{1}{S^2 - 2S + 2}$$



# Diagrama de Bode de polos imaginarios

$$\frac{1}{\left(1 + \left(\frac{s}{\omega_n}\right)^2\right)^n}$$

$$\frac{1}{\left(1 + \left(\frac{j\omega}{\omega_n}\right)^2\right)^n} = \begin{cases} \omega \rightarrow 0 & 1 = \begin{cases} \parallel = 0dB \\ \varphi = 0^\circ \end{cases} \\ \omega \rightarrow \infty & \left(\frac{j\omega}{\omega_n}\right)^{-2n} = \begin{cases} \parallel = \left(\frac{\omega}{\omega_n}\right)^{-2n} = -40n \log\left(\frac{\omega}{\omega_n}\right)dB \\ \varphi = -2n90^\circ = -n180^\circ \end{cases} \\ \omega \rightarrow \omega_n & \begin{cases} \omega \rightarrow \omega_n^- & \varphi = 0^\circ \\ \omega \rightarrow \omega_n^+ & \varphi = -2n90^\circ = -n180^\circ \end{cases} \end{cases}$$

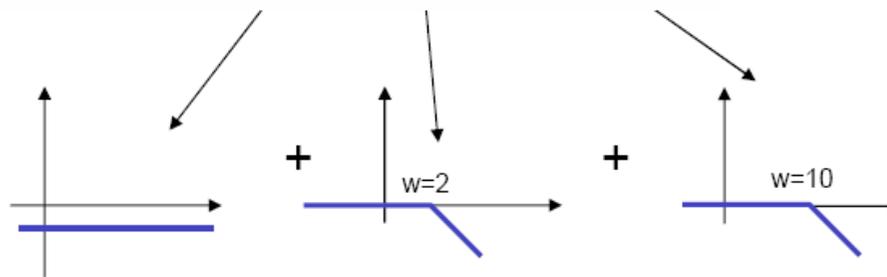


# DIAGRAMA DE BODE DE UN SISTEMA COMPLEJO

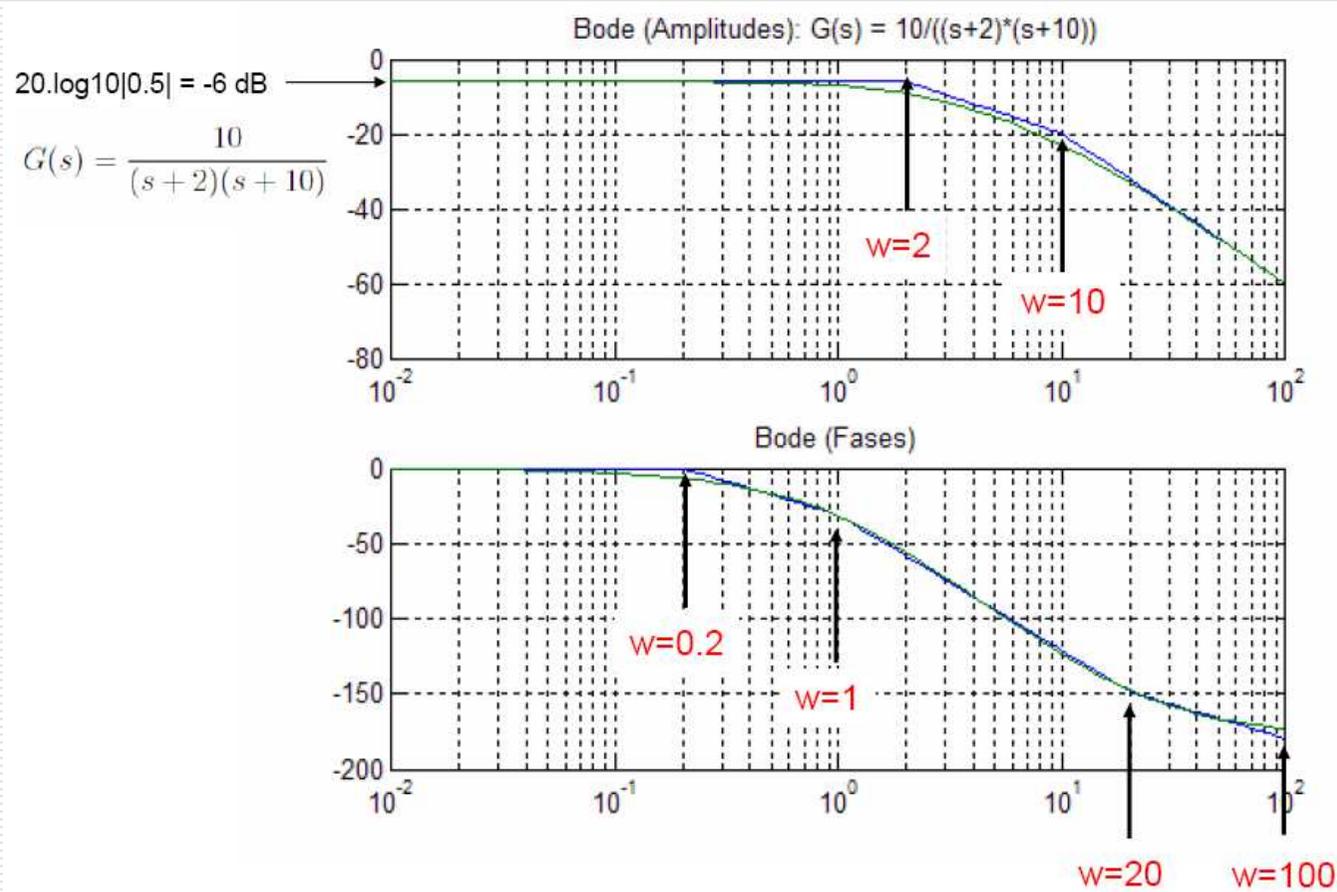
$$G(s) = \frac{10}{(s + 2)(s + 10)}$$

- Lo primero: factorizamos en bloques básicos (de Bodes conocidos)

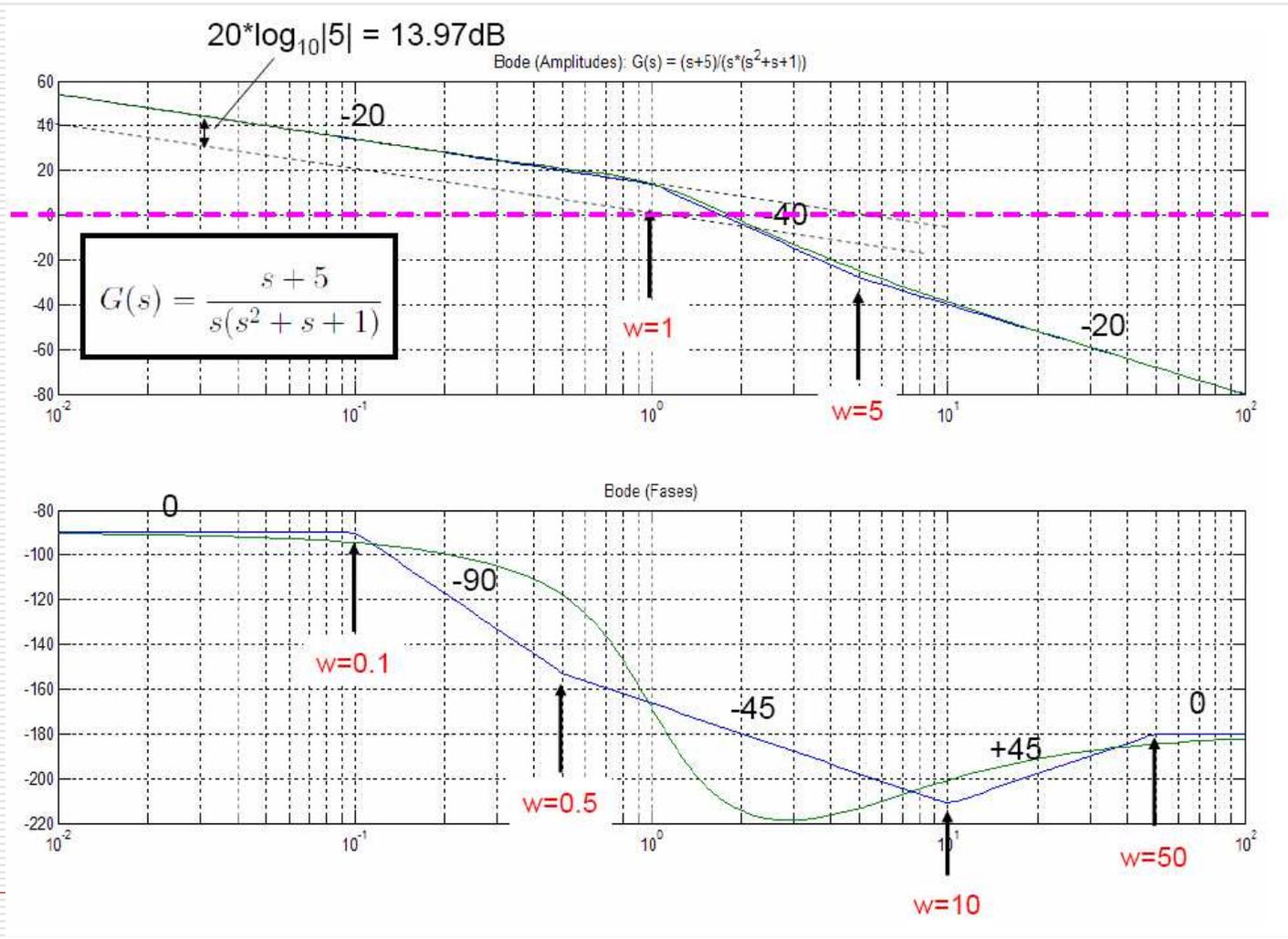
$$G(s) = \frac{10}{(s + 2)(s + 10)} = 0,5 \cdot \frac{2}{s + 2} \cdot \frac{10}{s + 10}$$



# DIAGRAMA DE BODE DE UN SISTEMA COMPLEJO



# Diagrama de Bode de un sistema: conjunto de polos y ceros



# Diagrama de Bode de un sistema: conjunto de polos y ceros

