# UNIT 1: Systems of linear equations.

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## Systems of linear equations with $m$ equations and $n$ unknowns.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\dots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

## Matrix notation: Augmented matrix of the system.

The size of the augmented matrix is $m \times (n+1)$.

$$(A|b) = \left(\begin{array}{cccc|c}
\multicolumn{4}{c}{\overbrace{\hphantom{a_{11} \quad a_{12} \quad \dots \quad a_{1n}}}^{\text{Coefficient matrix}}} & \\
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
\dots & \dots & \dots & \dots & \dots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m
\end{array}\right)$$

- as many rows as equations,
- as many columns as unknowns plus one.

## Elementary row operations (ERO)

- Adding to a given row $\mathbf{R_i}$ a scalar multiple of any other row $\mathbf{R_j}$: $\mathbf{R_i} + \mathbf{kR_j} \to \mathbf{R_i}$, with $k \in \mathbb{R}$.
- Exchanging any two rows: $\mathbf{R_i} \leftrightarrow \mathbf{R_j}$.
- Multiplying a given row $\mathbf{R_i}$ by a non-zero scalar: $\mathbf{kR_i} \to \mathbf{R_i}$, with $k \in \mathbb{R}$.

## Two matrices are row equivalent...

if there exists a sequence of ERO that transforms one into another.

## Two systems of linear equations are equivalent...

if their augmented matrices are row equivalent.

## The leading entry

of a nonzero row if the first entry from the left different from zero.

## A matrix is in row echelon form if:

- Each row with all entries equal to zero is below every row having at least one nonzero entry.
- The left most nonzero leading entry is to the right of the left most nonzero leading entry of the preceding row.
- All entries in a column below a leading entry are zero.

For example,

$$\begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Every matrix admits a row echelon form, but it is not unique.**

**A matrix is in reduced row echelon form if**

- it is in row echelon form,

and moreover it satisfies that:

- The leading entry in each nonzero row is 1.
- Each leading entry is the only nonzero entry in its column.

For example, $\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

**Every matrix has a unique reduced row echelon form.**

**The augmented matrix of a system in row echelon form is necessary to discuss the system, without solving it.**

- **Pivot position** corresponds with the position of the leading entry in the row echelon form matrix.
- **Pivot column** is a column that contains a pivot position.

## Gaussian elimination to obtain the row echelon form.

- The left most nonzero column of the matrix is the pivot column.
- Choose the pivot entry, a nonzero scalar, of the pivot column and place it in the pivot position. If necessary, interchange rows.
- Create zeros below the pivot position with ERO.
- Remove the row and the column that contains the pivot position. Repeat the same steps with the remaining matrix up to there is no more submatrices to be taken.

## Algorithm to obtain the reduced row echelon form.

Starting at a matrix in row echelon form:

- Select the pivot position that is in the right most pivot column.
- Create zeros above the pivot positions by ERO.
- If the pivot entry is not 1, divide by its value the entire row to obtain 1 in the pivot position.
- Repeat the same process to the left until there are not more pivot columns.

**The augmented matrix of the system in the reduced row echelon form allows to easily solve the system of linear equations.**

- **Basic variables** are in correspondance with pivot columns in the coefficient matrix $A$.
- **Free variables** are in correspondance with the nonpivot columns in the coefficient matrix $A$.

**Steps to solve the system by Gaussian elimination.**

1. Write the augmented matrix of the systems of linear equations.
2. Obtain a row echelon form of the augmented matrix.
3. Solve the system by back-substituion.

## Theorem (existence and uniqueness of solutions).

- A system is **consistent** if and only if the right most column in the augmented matrix is not a pivot column, that is, **if there are no rows such as**

$$(0, 0, \cdots, 0, c), \; c \neq 0$$

  in the augmented matrix in row echelon form.
- If the systems is consistent, there is
  - **Unique solution** if there are no free variables.
    The system is **consistent with a unique solution**.
  - **Infinitely many solutions** if there are free variables.
    The system is **consistent with infinitely many solutions**.

### Homogeneous systems: $Ax = 0_m$

- An homogeneous system is **always consistent** because it has the trivial solution $x = 0_n$.
- An homogeneous system is **consistent with infinitely many solutions** if it has **at least one free variable**.

### A system of linear equations is overdetermined...

if it has more equations than unknowns.

### A system of linear equations is underdetermined...

if it has less equations that unknowns.