UNIT 2: Matrices and determinants.

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A real matrix A of size $m \times n$

is a rectangular array of numbers $a_{ij} \in \mathbb{R}$ with m rows and n columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{i=1\dots m. \\ j=1\dots n.}}$$

The set of all matrices of size $m \times n$ is denoted by $M_{m\times n}$.

Particular matrices:

- $n = 1$. Column vector $\rightarrow V = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$.
- $m = 1$. Row vector $\rightarrow V = (v_1, \cdots, v_n)$.
- $m = n$. Square matrix.
- $m \neq n$. Rectangular matrix.

- Zero matrix $m \times n$:

$$0_{mn} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times n}$$

- Identity matrix of size n:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$$

The **main diagonal of a square matrix** $A = (a_{ij})$ consists of the elements $a_{11}, a_{22}, \ldots, a_{nn}$.

If A is a $n \times n$ square matrix,

A is **upper triangular** if $a_{ij} = 0$ whenever $i > j$.

$$\begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

A is **lower triangular** if $a_{ij} = 0$ whenever $i < j$.

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a_{nn} \end{pmatrix}.$$

If the only nonzero elements in a square matrix are the ones in the main diagonal, the matrix is said to be **diagonal**.

Addition of matrices of the same size $m \times n$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}.$$

Properties: For $A, B, C \in M_{m\times n}$

- Commutativity: $A + B = B + A$.
- Associativity: $(A + B) + C = A + (B + C)$.
- Identity element: $A + 0_{mn} = A = 0_{mn} + A$.
- Opposite element: $A + (-A) = 0_{mn}$.

Scalar multiplication by $\lambda \in \mathbb{R}$

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}.$$

Properties: For $A, B \in M_{m\times n}$ and $\lambda, \mu \in \mathbb{R}$,

- Distributivity with respect to scalars: $\lambda(A + B) = \lambda A + \lambda B$.
- Distributivity with respect to matrices: $(\lambda + \mu)A = \lambda A + \mu A$.
- Associativity: $\lambda\mu A = \lambda(\mu A)$.
- Identity element: $1A = A$.

Matrix product of A by a vector v

$$\begin{aligned} Av &= A\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{pmatrix} \\ &= v_1\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + v_2\begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + v_n\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \\ &= v_1A_1 + v_2A_2 + \cdots + v_nA_n. \end{aligned}$$

where A_i is the i-th column of A.

Then the multiplication Av is a linear combination of the columns of A.

Matrix multiplication of $A \in M_{m\times n}$ by $B \in M_{n\times p}$:

If we write the matrix $B = [B_1|B_2|\cdots|B_p]$ by columns,

$$AB = A[B_1|B_2|\cdots|B_p] = [AB_1|AB_2|\cdots|AB_p],$$

is a $m \times p$ matrix and $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Properties: For $A \in M_{m\times n}$, B, C matrices of suitable size and $\lambda \in \mathbb{R}$,

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $\lambda(AB) = (\lambda A)B = A(\lambda B)$
- $I_m A = A = AI_n$
- $A0_{np} = 0_{mp}, \quad 0_{rm}A = 0_{rn}$

Be careful: Matrix multiplication is not commutative: $AB \neq BA$.

Transpose matrix

Let $A \in M_{m \times n}$, the **transpose matrix of** A is a matrix of size $n \times m$ such that its columns are the rows of A. The transpose matrix of A is denoted by A^T.

Example: If $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 6 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 2 & 6 \end{pmatrix}$.

Properties: For matrices A, B of suitable size and $\lambda \in \mathbb{R}$,

- $(A^T)^T = A$.
- $(A + B)^T = A^T + B^T$.
- $(\lambda A)^T = \lambda A^T$.
- $(AB)^T = B^T A^T$. **Note the change in the order of matrices.**

A matrix A is...

- **symmetric** if $A^T = A$. (Then A must be square).
- **skewsymmetric** if $A^T = -A$. (Then A must be square).
- **orthogonal** if $A^T A = AA^T = I$. (A is not necessarily square).

A square matrix A of size $n \times n$ has inverse...

if there exists a $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = \mathrm{I}_n.$$

The matrix A^{-1} denotes the inverse matrix of A.

If A is invertible,

the systems of linear equations $Ax = b$ has a unique solution and it is given by $x = A^{-1}b$.

Properties: For invertible matrices A and B of suitable size,

- $(A^{-1})^{-1} = A$.
- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$.
- The inverse matrix is unique, if it exists.

How to compute the inverse matrix by elementary row operations (ERO):

$$\begin{pmatrix} A & | & I_n \end{pmatrix} \underset{\text{By ERO}}{\sim} \begin{pmatrix} I_n & | & A^{-1} \end{pmatrix}.$$

The identity matrix has the same size as the matrix A.

Determinants

A scalar can be assigned to any square matrix. That scalar is called **determinant**.

How to compute determinants of matrices $A \in M_{n\times n}$ with $n = 2$ and $n = 3$. (Rule of Sarrus)

- $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\det A = |A| = a_{11}a_{22} - a_{12}a_{21}$.
- $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

For a square matrix $A \in M_{n\times n}$:

- The **determinant associated with** a_{ij} is the determinant of the matrix A_{ij} of size $(n-1)\times(n-1)$ obtained by removing the i-th row and the j-th column of the matrix A.

  Example: $A = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\det A_{21} = \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 0 - 3 = -3.$

- The (i,j)-**cofactor** C_{ij} is the value $(-1)^{i+j}\det A_{ij} = C_{ij}$.
  In the previous example, $C_{21} = (-1)^{2+1}\det A_{21} = -(-3) = 3.$

- The **adjugate matrix of** A is the matrix of size $n \times n$ that consists of the (i,j)-cofactor of all the elements in A:

$$\operatorname{Adj} A = \begin{pmatrix} C_{11} & \dots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \dots & C_{nn} \end{pmatrix}.$$

The determinant of any $n \times n$ square matrix A is computed as:

- a cofactor expansion across the i-th row:

$$\det A = |A| = a_{i1}C_{i1} + \cdots + a_{in}C_{in},$$

- or a cofactor expansion across the j-th column:

$$\det A = |A| = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}.$$

Usually we choose the row or column with more zeroes to compute the determinants as a cofactor expansion across it to simplify the computations.

Properties of the determinants for $A \in M_{n\times n}$:

Associated with elementary row operations:

- If $A \overset{\lambda R_i + R_j \to R_j}{\sim} B$, then $\mathrm{detA} = \mathrm{det} B$.
- If $A \overset{R_i \leftrightarrow R_j}{\sim} B$, then $\mathrm{detA} = -\mathrm{det} B$.
- If $A \overset{\lambda R_j \to R_j}{\sim} B$, then $\mathrm{detB} = \lambda \mathrm{det} A$.

Other properties:

- If two rows are equal or proportional, $\mathrm{det} A = 0$.
- If a matrix has a row of zeroes, $\mathrm{det} A = 0$.
- If a matrix is invertible, $\text{det}A \neq 0$.
- If a matrix is upper or lower triangular or diagonal, then $\mathrm{det} A = a_{11} \ldots a_{nn}$.
- $\det(A^{-1}) = \dfrac{1}{\det A}$.
- $\det(\lambda A) = \lambda^n \det A$.
- $\det(I_n) = 1$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(A^T) = \det(A)$.

Application of determinants:

- **To compute the inverse matrix:** $A^{-1} = \dfrac{1}{\det A}(\mathrm{AdjA})^T$.

  Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 6 \end{pmatrix}$, $\det A = 1$, $\mathrm{Adj}\, A = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$.

  Then, $A^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$.

- **Cramer's rule:** The solution of the system $Ax = b$ if $\det A \neq 0$ is given by

$$x_i = \frac{\det A_i(b)}{\det A}, \text{where } A_i(b) = (A_1 \ldots A_{i-1}\, b\, A_{i+1} \ldots A_n)$$

  and A_i denotes the i-th column of the matrix A.

Example of how to solve the systems by Cramer's rule.

Given the following system $\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

In this case $A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

As $\det A = 12 - 10 = 2 \neq 0$, we can use Cramer's rule to compute the solution of the system $Ax = b$.

The matrices we need are

$$A_1(b) = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}, \quad A_2(b) = \begin{pmatrix} 3 & 6 \\ -5 & 8 \end{pmatrix}.$$

Then,

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{24+16}{2} = 20, \; x_2 = \frac{\det A_2(b)}{\det A} = \frac{24+30}{2} = 27.$$

Characterization of invertible matrices.

Let $A \in M_{n \times n}$. The following statements are equivalent:

1. A is invertible.
2. A is a row equivalent matrix to the identity matrix I_n.
3. A has n pivot positions.
4. A has n pivot columns.
5. $\det A \neq 0$.
6. The systems of linear equations $Ax = 0$ has a unique solution, the trivial one $x_n = 0_n$.
7. For each $b \in \mathbb{R}^n$ the system of linear equations $Ax = b$ has a unique solution.