UNIT 3: Real vector spaces.

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A real vector space

is a nonempty set whose elements are called **vectors** and the following operations are defined:

- **addition of vectors** ($u + v \in V$ for all $u, v \in V$) that satisfies the following properties
  1. Commutativity: $u + v = v + u$.
  2. Associativity: $(u + v) + w = u + (v + w)$.
  3. Identity element $u + 0 = 0$.
  4. Opposite element $u + (-u) = 0$.
- **scalar multiplication** ($\lambda\, v \in V$ for all $v \in V$, $\lambda \in \mathbb{R}$) that satisfies the following properties
  1. Distributivity with respect to scalars: $\lambda(u + v) = \lambda u + \lambda v$.
  2. Distributivity with respect to vectors: $(\lambda + \mu)u = \lambda u + \mu u$.
  3. Associativity: $\lambda(\mu u) = (\lambda\mu)u$.
  4. Identity element $1u = u$.

for all $u, v, w \in V$, $\lambda, \mu \in \mathbb{R}$.

Example of vector spaces:

- $\mathbb{R}^n$ for every $n \in \mathbb{N}$.
- $M_{m\times n}$, that is, the set of matrices with the same size.
- The set of polynomials with degree less or equal to n.

A subspace of a vector space V is...

a set H of V that satisfies the following three properties:

1. $0 \in H$,
2. if $u, v \in H$, then $u + v \in H$,
3. if $u \in H$, $\lambda \in \mathbb{R}$, then $\lambda u \in H$.

- Any vector space is a subspace.
- Any subspace is a vector space.

Distinguished subspaces:

- The **zero subspace** $H = \{0\}$. It only contains the identity element with respect to the addition.
- The **total subspace** $H = V$.

Other examples of subspaces:

- The **only subspaces in** $\mathbb{R}^2$ are the zero subspace, the total subspace and all the straight lines that contain the origin.
- The **only subspaces in** $\mathbb{R}^3$ are the zero subspace, the total subspace, the straight lines that contain the origin and all the planes that contain the origin.

A vector $y \in V$ is a linear combination of a set of vectors $\{v_1, \ldots, v_p\}$

if there exist coefficients $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ such that

$$y = \lambda_1 v_1 + \ldots \lambda_p v_p.$$

The set spanned by $\{v_1, \ldots, v_p\}$, $\operatorname{span}\{v_1, \ldots, v_p\}$,

is the set of all the linear combinations of $\{v_1, \ldots, v_p\}$.

The set $H = \operatorname{span}\{v_1, \ldots, v_p\}$ is always a vector subspace.

The set of vectors $\{v_1, \ldots, v_p\}$ is the **spanning set for** H.

The null subspace of a matrix $A \in M_{m\times n}$ is

$$\operatorname{Nul} A = \{x \in \mathbb{R}^n \mid Ax = 0_m\}.$$

That is, $\operatorname{Nul} A$ is the solution set of the homogeneous system $Ax = 0$.

$\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$

where n is the number of columns of A.

Method to compute $\operatorname{Nul} A$

1. Solve the system of linear equations $Ax = 0$ by obtaining the row echelon form of the augmented matrix of the system.
2. Write the solution of the system in parametric form.
3. Obtain the vectors in the spanning set for the solution set.

Remark: $\operatorname{Nul} A$ has **as many vectors in a spanning set as number of free variables of the system** $Ax = 0$.

The column space of a matrix $A \in M_{m\times n}$ is

$$\mathrm{Col}\, A = \{b \in \mathbb{R}^m \mid Ax = b \text{ for some } x \in \mathbb{R}^n\}.$$

$\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^m$

where m is the number of rows of A.

Method to compute $\mathrm{Col}\, A$

1. Obtain a matrix in row echelon form row equivalent to A.
2. Identify the pivot columns of the matrix in row echelon form.
3. The spanning set of $\mathrm{Col}\, A$ is given by the columns corresponding to the pivots columns, but in the given matrix A.

Remark: $\mathrm{Col}\, A$ has **as many vectors in the spanning set as the number of pivot columns in** A.

A set of nonzero vectors $\{v_1, \dots, v_p\}$ is LINEARLY INDEPENDENT if

the unique solution of the vector equation $\lambda_1 v_1 + \cdots + \lambda_p v_p = 0$ is the trivial one, that is, $\lambda_1 = \cdots = \lambda_p = 0$.
That is, the system whose augmented matrix is $(v_1 \dots v_p \dots 0)$ and the unknowns are $\lambda_1, \dots, \lambda_p$ is **consistent with unique solution**.

A set of vectors $\{v_1, \dots, v_p\}$ is LINEARLY DEPENDENT if

the vector equation $\lambda_1 v_1 + \cdots + \lambda_p v_p = 0$ has infinitely many solutions.
That is, the system whose augmented matrix is $(v_1 \dots v_p \dots 0)$ and the unknowns are $\lambda_1, \dots, \lambda_p$ is **consistent with infinitely many solutions**.

A set of vectors that contains the zero vector is linearly dependent.

$B = \{b_1, \ldots, b_p\}$ is a basis for the subspace H of V if

- B is a linearly independent set (l.i.).
- B is a spanning set for H: $H = \mathrm{span}\{b_1, \ldots, b_p\}$.

The standard basis for $\mathbb{R}^n$

is the set of vectors given by the columns of the identity matrix $I_n = (e_1 | \ldots | e_n)$.

A basis $B = \{b_1, \ldots, b_p\}$ for H is

- the smallest spanning set of H,
- and the largest set of linearly independent vectors in H.

Dimension of a subspace H

is the number of vectors that a basis $B = \{b_1, \ldots, b_p\}$ for H has. $\mathbf{\dim H = p}$.

Spanning set theorem.

Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in V,
$H = \mathrm{span}\,\{v_1, \ldots, v_p\}$.

- If a vector v_k in S is a linear combination of the remaining vectors, then

$$\mathrm{span}\{S - \{v_k\}\} = \mathrm{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_p\} = H.$$

- If $H \neq \{0\}$, a subset of S is a basis for H.

Example:

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
is l.i., but

it does not span $\mathbb{R}^2$.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$
is a basis for $\mathbb{R}^2$.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$
spans $\mathbb{R}^2$,

but they are not l.i.

The rank of a matrix $A \in M_{m \times n}$

is equal to the number of pivot positions in A.

Note that:

- $\operatorname{rank} A = \dim \operatorname{Col} A$.
- $\operatorname{rank} A \leq \min\{m, n\}$.

Rank theorem.

If $A \in M_{m \times n}$,

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n \text{ (number of columns in } A\text{)}.$$

Coordinates of a vector x in the basis $B = \{b_1, \dots, b_p\}$.

$$[x]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}_B = x_1 b_1 + \cdots + x_p b_p.$$

Change-of-basis matrix.

Let two bases $B = \{b_1, \dots, b_p\}$ and $C = \{c_1, \dots, c_p\}$, the change of coordinates of a vector in the basis B to the basis C is given by

$$[x]_C = P_{CB}[x]_B,$$

where $P_{CB} = \begin{pmatrix} [b_1]_C & \dots & [b_p]_C \end{pmatrix}$
is the **change-of-basis matrix from** B **to** C.

$(P_{CB})^{-1} = P_{BC}$

A linear transformation T

from a vector space V to another vector space W assigns a vector $v \in V$ to a unique vector $T(v)$ in W in such a way that

1. $T(0) = 0$,
2. $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
3. $T(\lambda u) = \lambda T(u)$ for all $\lambda \in \mathbb{R}$, $u \in V$.

- The **kernel of** T is the set of all vectors v in V such that $T(v) = 0$.
- The **image of** T is the set of all vectors w in W such that $w = T(v)$ for some v in V.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation,

then T has an associated matrix $A \in M_{m \times n}$ such that $T(x) = Ax$, where $A = \begin{pmatrix} T(e_1) & \dots & T(e_n) \end{pmatrix}$.

Moreover: $\quad \mathbf{ker\,T} = \mathbf{Nul\,A}, \quad \mathbf{Im\,T} = \mathbf{Col\,A}$.

Examples:

- $T_1(x, y, z) = (1, x + y, 3z)$ is NOT a linear transformation because $T(0, 0, 0) = (1, 0, 0) \neq (0, 0, 0)$.
- $T_2(x, y) = (x^2 + y, y)$ is NOT a linear transformation because the properties 2 and 3 are not satisfied.
- $T_3(x, y) = (2x + 3y, -x + y, 4y)$ IS a linear transformation with an associated matrix of size 3×2 because $T_3 \colon \mathbb{R}^2 \to \mathbb{R}^3$.

$$T_3 = \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 4 \end{pmatrix}, \quad T_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ -x + y \\ 4y \end{pmatrix} = T_3(x, y)^T.$$

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is...

- **injective** if the only vector $x \in \mathbb{R}^n$ such that $T(x) = 0$ is the zero vector $x = 0$.

  Equivalently, if all the columns of A are linearly independent.

- **surjective or onto** if each vector $b \in \mathbb{R}^m$ is the image of some vector $x \in \mathbb{R}^n$, that is, $T(x) = b$.

  Equivalently, if all the columns of A span $\mathbb{R}^m$.

- **one-to-one** if it is injective and onto.