UNIT 4: Eigenvalues and eigenvectors. Diagonalization.

María Barbero Liñán

Universidad Carlos III de Madrid
Bachelor in Statistics and Business

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Let A be a square matrix, $A \in M_{n \times n}$.

An **eigenvector of** A is a nonzero vector x such that

$$Ax = \lambda x \text{ for some } \lambda \in \mathbb{R}.$$

A scalar λ is an **eigenvalue of** A if there exists a nontrivial solution x of $Ax = \lambda x$.

The vector x is an **eigenvector of** A corresponding with λ.

To find the eigenvalues of A:

1. Find the **characteristic polynomial of** A given by $P_A(\lambda) = \det(A - \lambda \mathrm{I})$.
2. Find the roots of the polynomial, that is, find λ in the equation $P_A(\lambda) = 0$.

To find the eigenvectors of A corresponding to the eigenvalue λ:

- Find the nullspace of the matrix $A - \lambda \mathrm{I}_n$.

The vector space of eigenvectors of the eigenvalue λ is denoted by $V(\lambda)$. Then,

$$V(\lambda) = \mathrm{Nul}(A - \lambda \mathrm{I}_n).$$

If the real number 0 is an eigenvalue of A,

$$\det (A - 0\mathrm{I}_n) = \det A = 0.$$

Then A is not invertible.

For each eigenvalue we define:

- **the algebraic multiplicity** n_λ as the multiplicity of λ as a root of the characteristic polynomial.
- **the geometric multiplicity** m_λ as the dimension of the subspace of eigenvectors of eigenvalue λ, $m_\lambda = \dim V(\lambda)$.

It is always true that $m_\lambda \leq n_\lambda$.

A square matrix $A \in M_{n\times n}$ is diagonalizable...

if there exist an invertible matrix P and a diagonal matrix D, both with size $n \times n$, such that

$$A = PDP^{-1}.$$

It is said that A is **similar** to D.

The following statements are equivalent:

1. A is diagonalizable.
2. A has n linearly independent eigenvectors.
3. $\sum_{i=1}^{k} n_{\lambda_i} = n$ y $n_{\lambda_i} = m_{\lambda_i}$ for each $i = 1, \ldots, k$, where k is the number of different eigenvalues of A.
4. There exists a basis of $\mathbb{R}^n$ that consists only of eigenvectors.

Theorem.

Thel **set of eigenvectors** $\{v_1, \ldots, v_p\}$ **corresponding to different eigenvalues** $\lambda_1, \ldots, \lambda_p$ of a $n \times n$ matrix A **is linearly independent**.

Consequences of the theorem:

If A has n different eigenvalues, then A is diagonalizable.

Steps to diagonalize a matrix:

1. Find the eigenvalues of A by solving $\det(A - \lambda \mathrm{I}_n) = 0$.
   - From the linear decomposition of the characteristic polynomial we obtain the algebraic multiplicity of each eigenvalue.
2. Find the eigenvectors corresponding to each eigenvalue, that is, find the spanning set of $\mathrm{Nul}(A - \lambda \mathrm{I}_n)$.
   - The dimension of $\mathrm{Nul}(A - \lambda \mathrm{I}_n)$ gives the geometric multiplicity of the eigenvalue.
3. Determine if the matrix is diagonalizable or not by using the equivalent statements in the previous slide.

Steps to diagonalize a matrix:

4 If A is diagonalizable, find the matrices P and D.

- The columns of P are the eigenvectors.
- The entries in the main diagonal of D are the eigenvalues of A, in the same order as the corresponding eigenvectors in P.

Example: $P = (v_{\lambda_1} v_{\lambda_2} \dots v_{\lambda_n})$, $\quad D = \begin{pmatrix} \lambda_1 & \dots & 0 \\ 0 & \lambda_2 \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}$.

5 Check that $A = PDP^{-1}$.

Example: Diagonalize the following matrix.

$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

1. $P_A(\lambda) = |A - \lambda \mathrm{I}_2| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 21.$

   From $P_A(\lambda) = 0$ we obtain two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -7$. Then $n_{\lambda_1} = n_{\lambda_2} = 1$.

2. $\mathrm{Nul}(A - \lambda_1 \mathrm{I}) = \mathrm{Nul} \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} = \mathrm{span} \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} = V(3).$

   $\mathrm{Nul}(A - \lambda_2 \mathrm{I}) = \mathrm{Nul} \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} = \mathrm{span} \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\} = V(-7).$

   The geometric multiplicity of both eigenvalues is $1 = m_{\lambda_1} = m_{\lambda_2}$.

Example: Diagonalize the following matrix.

3 The matrix A is diagonalizable because $n_{\lambda_1} + n_{\lambda_2} = 2$, $n_{\lambda_1} = m_{\lambda_1} = 1$, $n_{\lambda_2} = m_{\lambda_2} = 1$.

4 $P = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$.

5

$$\begin{aligned} A = PDP^{-1} &= (PD)P^{-1} = \begin{pmatrix} 9 & -7 \\ 3 & 21 \end{pmatrix} \frac{1}{-10} \begin{pmatrix} -3 & -1 \\ -1 & 3 \end{pmatrix} \\ &= \frac{1}{-10} \begin{pmatrix} 20 & -30 \\ -30 & 60 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}. \end{aligned}$$