MATHEMATICAL METHODS II
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Homework sheet 1: SYSTEMS OF LINEAR EQUATIONS
(with solutions)
Year 2011-2012

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1. Express in your own words the next elementary row operation to perform in order to continue with the solving process of the following linear systems.

$$\text{a)}\quad \begin{array}{rrrrl} x_1 & +4x_2 & -2x_3 & +8x_4 = & 12 \\ & x_2 & -7x_3 & +2x_4 = & -4 \\ & & 5x_3 & -x_4 = & 7 \\ & & x_3 & +3x_4 = & -5 \end{array} \qquad \text{b)}\quad \begin{array}{rrrrl} x_1 & -3x_2 & +5x_3 & -2x_4 = & 0 \\ & x_2 & +8x_3 & = & -4 \\ & & 2x_3 & = & 3 \\ & & & x_4 = & 1 \end{array}$$

Solution: (a) The system is not in row echelon form yet. In order to obtain the system in row echelon form the fourth equation must be replaced by the linear combination of the third one minus 5 times the fourth equation, that is, $\mathrm{Eq}_3 - 5\mathrm{Eq}_4 \to \mathrm{Eq}_4$.

(b) The system is already in row echelon form, but not in reduced row echelon form. To continue the process to obtain the system in reduced row echelon form the second equation must be replaced by the second equation minus four times the third equation, that is, $\mathrm{Eq}_2 - 4\mathrm{Eq}_3 \to \mathrm{Eq}_2$.

2. Consider the following matrix to be an augmented matrix of a linear system. Express in your own words the next elementary row operation to perform in the solving process of the system.

$$\begin{bmatrix} 1 & 8 & 2 & -7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \end{bmatrix}$$

Solution: Add to row 3 the row 2 multiplied by -2, that is, $-4R_2 + R_3 \to R_3$.

3. Determine the consistency of the linear system whose augmented matrix has been transformed into the following matrix by elementary row operations.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

Solution: If the last row is rewritten as an equation, we have $5x_3 = 0$. Hence $x_3 = 0$. From here we can obtain the values of x_1 and x_2 by back substitution. Thus the system is consistent.

4. Check if $(3, 4, -2)$ is a solution of the following system.

$$\begin{array}{rrrl} 5x_1 & -x_2 & +2x_3 = & 7 \\ -2x_1 & +6x_2 & +9x_3 = & 0 \\ -7x_1 & +5x_2 & -3x_3 = & -7 \end{array}$$

Solution: Replace $x_1 = 3$, $x_2 = 4$ and $x_3 = -2$ in the given system.

$$\begin{array}{rrrrrl} 5\cdot 3 & -4 & +2(-2) = & 15-4-4 = & 7 & \text{OK.} \\ -2\cdot 3 & +6\cdot 4 & +9(-2) = & -6+24-18 = & 0 & \text{OK.} \\ -7\cdot 3 & +5\cdot 4 & -3(-2) = & -21+20+6 = & 25 \neq & -7. \end{array}$$

The last equation is not satisfied. Then $(3, 4, -2)$ is not a solution of the given system.

5. Assume that the augmented matrix of a linear system has been transformed into the following matrix by elementary row operations. Continue the solving process and describe the solution set of the original system.

$$\text{a)} \begin{bmatrix} 1 & 2 & 0 & -3 & -9 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{b)} \begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \text{c)} \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Solution: (a) The solution set is $S = \{(-6, 6, 5, -1)\}$.

(b) It is impossible that $0x_3 = 1$, hence the system is inconsistent. The solution set is empty $S = \emptyset$.

(c) The solution set is $S = \{(0, 0, 0)\}$.

6. Solve the following linear systems:

$$\text{a)} \left\{ \begin{array}{rcr} x_1 - 5x_2 + 4x_3 & = & -3 \\ 2x_1 - 7x_2 + 3x_3 & = & -2 \\ -2x_1 + x_2 + 7x_3 & = & -1 \end{array} \right. \qquad \text{b)} \left\{ \begin{array}{rrrr} & x_2 & +4x_3 = & -5 \\ x_1 & +3x_2 & +5x_3 = & -2 \\ 3x_1 & +7x_2 & +7x_3 = & 6 \end{array} \right.$$

$$\text{c)} \left\{ \begin{array}{rrrr} x_1 & & -3x_3 = & 8 \\ 2x_1 & +2x_2 & +9x_3 = & 7 \\ & x_2 & +5x_3 = & -2 \end{array} \right.$$

Solution: (a) $S = \{(-3, -5, 6, 3)\}$. (b) Inconsistent, $S = \emptyset$. (c) $S = \{(5, 3, -1)\}$.

7. Study if the following systems are consistent:

$$\text{a)} \left\{ \begin{array}{rrrrcr} x_1 & & -2x_3 & & = & -1 \\ & x_2 & & -x_4 & = & 2 \\ & -3x_2 & +2x_3 & & = & 0 \\ -4x_1 & & & +7x_4 & = & -5 \end{array} \right. \qquad \text{b)} \left\{ \begin{array}{rrrrr} x_1 & & +3x_3 & = & 2 \\ & x_2 & & -3x_4 = & 3 \\ & -2x_2 & +3x_3 & +2x_4 = & 1 \\ 3x_1 & & & +7x_4 = & -5 \end{array} \right.$$

$$\text{c)} \left\{ \begin{array}{rrrrr} x_1 & & & -2x_4 = & -3 \\ & 2x_2 & +2x_3 & = & 0 \\ & & x_3 & +3x_4 = & 1 \\ -2x_1 & +3x_2 & +2x_3 & +x_4 = & 5 \end{array} \right.$$

Solution: (a) The system is consistent with a unique solution. (b) The system is consistent with a unique solution. (c) The system is consistent with infinitely many solutions.

8. Determine the value(s) of h such that the following augmented matrices describe consistent linear systems.

$$\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix} \quad \begin{bmatrix} 2 & -6 & -3 \\ -4 & 12 & h \end{bmatrix}$$

Solution: The system on the left-hand side is consistent for any $h \in \mathbb{R}$. It has a unique solution if $h \neq 12$ and it has infinitely many solutions if $h = 12$.

The system on the right-hand side is consistent if and only if $h = 6$. For $h = 6$ the system has infinitely many solutions.

9. Determine the values of h and k such that the following system is consistent.

$$\begin{aligned} 2x_1 \quad -x_2 &= h \\ -6x_1 \quad +3x_2 &= k \end{aligned}$$

Solution: The system is consistent if and only if for h, $k \in \mathbb{R}$ such that $3h + k = 0$.

10. Do the three straight lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, $-x_1 - 3x_2 = 4$ have the same intersection point? Explain your answer.

Solution: If we compute the row echelon form of the system defined by the equations of the three straight lines, $\begin{pmatrix} 1 & -4 & 1 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \end{pmatrix}$, we see that none of the rows has a pivot in the last column. Thus the system is consistent. As the system has two pivots, as many as unknowns, the system is consistent with a unique solution. In other words, the three straight lines have the same intersection point.

11. Find the elementary row operation that transforms the first matrix into the second one and in the other way round.

$$\text{a)} \begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix}$$

$$\text{b)} \begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$$

$$\text{c)} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 3 & 5 & 1 \\ 3 & -4 & 7 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 3 & 5 & 1 \\ 0 & 2 & 7 & -7 \end{bmatrix}$$

$$\text{d)} \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Solution: (a) Interchange row 1 and row 2.

(b) From the first matrix into the second one, scale row 2 by $-1/2$. From the second matrix to the first one, scale row 2 by -2.

(c) From the first matrix into the second one, $-3R_1 + R_3 \to R_3$. From the second matrix to the first one, $R_3 + 3R_1 \to R_3$.

(d) From the first matrix into the second one, $3R_2 + R_3 \to R_3$. From the second matrix to the first one, $R_3 - 3R_2 \to R_3$.

12. Give three different augmented matrices whose solution sets only contains $x_1 = -2$, $x_2 = 1$, $x_3 = 0$.

Solution: The three given equations determine the following augmented matrix $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Now by applying elementary row operations we can obtain infinite augmented matrices. For instance,

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix},$$

come from $R_2 + R_3 \to R_2$, $R_1 + R_3 \to R_3$, respectively.

13. Determine which matrices are in reduced echelon form, which others are in echelon form (but not reduced) and which others are not in row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: Reduced Echelon Form, Echelon Form, No Echelon Form, Reduced Echelon Form.

14. Row reduce the following matrices to reduced echelon form, determine the pivot columns and the pivots in both the original matrices and in the final matrices.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution: The pivots are the bold numbers and the pivot columns are those that contain the pivots.

$$\begin{bmatrix} \mathbf{1} & 1 & 1 \\ 0 & \mathbf{2} & 2 \\ 0 & 0 & \mathbf{3} \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & 4 & 7 & 10 \\ 2 & \mathbf{5} & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & -1 & 2 \\ 0 & \mathbf{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & 0 & 1 & 0 & 0 \\ 0 & \mathbf{0} & 1 & 0 & 1 \\ 0 & 1 & \mathbf{0} & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & -1 \\ 0 & \mathbf{1} & 0 & 0 & -1 \\ 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 1 \end{bmatrix}$$

15. Find the general solution of linear systems whose augmented matrices are the following ones:

$$\text{a)}\begin{bmatrix} 1 & 5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{b)}\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{c)}\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix} \qquad \text{d)}\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ 6 & 8 & -4 & 0 \end{bmatrix}$$

Solution: (a) Inconsistent, $S = \emptyset$.

(b) Consistent with infinitely many solutions, $S = \{(x_1, x_2, x_3, x_4, x_5) \,|\, x_1 = 5 + 3x_5,\ x_2 = 1 + 4x_5,\ x_4 = 4 - 9x_5,\ x_4, x_5 \in \mathbb{R}\}$.

(c) Consistent with infinitely many solutions, $S = \{(x_1, x_2, x_3) \,|\, x_1 = 4 + 5x_3,\ x_2 = 5 + 6x_3,\ x_3 \in \mathbb{R}\}$.

(d) Consistent with infinitely many solutions, $S = \{(x_1, x_2, x_3) \,|\, x_1 = 0,\ x_2 = \frac{x_3}{2},\ x_3 \in \mathbb{R}\}$.

16. Determine the values of h and k such that the following linear system (a) has no solutions, (b) has a unique solution and (c) has infinitely many solutions.

$$\begin{aligned} x_1 + hx_2 &= 1 \\ 2x_1 + 3x_2 &= k \end{aligned}$$

Solution: (a) is fulfilled for $h = \frac{3}{2}$ and $k \neq 2$.

(b) is fulfilled for $h \neq \frac{3}{2}$ and $k \in \mathbb{R}$.

(c) is fulfilled for $h = \frac{3}{2}$ and $k = 2$.

17. Suppose that a system of linear equations has a 3×5 coefficient matrix with three pivot columns. Is the system consistent? Why or why not?

Solution: The system is consistent with infinitely many solutions because it has 3 basic variables and 1 free variable. As there are no rows enough such that one has a pivot position in the last column, the system is always consistent.

18. A system with more equations than unknowns is called *overdetermined*. One with more unknowns than equations is called *underdetermined*. Do consistent overdetermined systems exist? Reason your answer and provide an example if they do exist. Is it possible to find an underdetermined linear system that has a unique solution?

Solution: Consistent overdetermined systems do exist. For instance,

$$\left\{ \begin{array}{cccc} x_1 & & & = 0, \\ & x_2 & & = 0, \\ & & x_3 & = 1, \\ x_1 & +x_2 & +x_3 & = 1. \end{array} \right.$$

Undetermined linear systems with a unique solution do not exist. When we reduce the system to row echelon form, there are less pivots than variables. Thus there are always free variables and there are no rows enough such that one has a pivot position in the last column as in the example in exercise 17.

19. Reason if the following statements are true or false. (To reason means that theorems or suitable results will be used to prove the validity of the statements and counterexamples must be provided if any statement is false).

1. If a matrix B is obtained from another matrix A by means of elementary row operations, then A can be obtained from B by means of elementary row operations.
2. Each matrix is row equivalent to a unique matrix in row echelon form.
3. To multiply all the elements in a row by a constant is an elementary row operation.
4. If the augmented matrices of two systems of equations are row equivalent, then both systems have the same solution set.
5. Any system with n linear equations and n unknowns has at most n solutions.
6. If a system of linear equations has two different solutions, then it has infinitely many solutions.
7. If a system of linear equations does not have free variables, then it has a unique solution.
8. An overdetermined system cannot have a unique solution.
9. An underdetermined system cannot have a unique solution.
10. If all the columns of the coefficient matrix of a consistent system are pivot columns, then the systems has a unique solution.
11. A system of linear equations has infinitely many solutions if and only if at least one column in the coefficient matrix does not contain a pivot position
12. A consistent system of linear equations has infinitely many solutions if and only if at least one column in the coefficient matrix does not contain a pivot position.
13. An inconsistent system of linear equations sometimes has a unique solution.
14. A 5×7 matrix cannot have a pivot position in every row.
15. A 6×5 matrix cannot have a pivot position in every row.

Solution: 1-True, 2-False, 3-True, 4-True, 5-False, 6-True, 7-False, 8-False, 9-True, 10-True, 11-False, 12-True, 13-False, 14-False, 15-True.

20. Find a polynomial of degree 2, $p(t) = a_0 + a_1 t + a_2 t^2$, whose graph passes through the following points in the plane: $(1, 12), (2, 15), (3, 16)$. Such a polynomial is called an *interpolating polynomial* for those points.

Solution: Having in mind that $p(1) = 12$, $p(2) = 15$, $p(3) = 16$, we obtain a system of three equations and three unknowns given by a_0, a_1 and a_2. After computations $p(t) = 7 + 6t - t^2$.

21. Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \end{aligned}$$

Solution: The system is consistent with infinitely many solutions. Thus the two planes intersect. The intersection is given by $\{(-3x_3, 2x_3, x_3) \mid x_3 \in \mathbb{R}\}$.

22. Determine if the following systems have a nontrivial solution. Try to use as few row operations as possible.

a) $$\begin{array}{rrrl} 2x_1 & -5x_2 & +8x_3 = & 0 \\ -2x_1 & -7x_2 & +x_3 = & 0 \\ 4x_1 & +2x_2 & +7x_3 = & 0 \end{array}$$

b) $$\begin{array}{rrrl} x_1 & -3x_2 & +7x_3 = & 0 \\ -2x_1 & +x_2 & -4x_3 = & 0 \\ x_1 & +2x_2 & +9x_3 = & 0 \end{array}$$

c) $$\begin{array}{rrrl} -3x_1 & +5x_2 & -7x_3 = & 0 \\ -6x_1 & +7x_2 & +x_3 = & 0 \end{array}$$

d) $$\begin{array}{rrrl} -5x_1 & +7x_2 & +9x_3 = & 0 \\ x_1 & -2x_2 & +6x_3 = & 0. \end{array}$$

Solution: (a), (c) and (d) have nontrivial solutions. (b) does not have nontrivial solutions.

23. Write the solution set of the given homogeneous systems in parametric vector form.

a) $$\begin{array}{rrrl} x_1 & +3x_2 & +x_3 = & 0 \\ -4x_1 & -9x_2 & +2x_3 = & 0 \\ & -3x_2 & -6x_3 = & 0 \end{array}$$

b) $$\begin{array}{rrrl} x_1 & +3x_2 & -5x_3 = & 0 \\ x_1 & +4x_2 & -8x_3 = & 0 \\ -3x_1 & -7x_2 & +9x_3 = & 0 \end{array}$$

Solution:

(a) $\left\{\begin{array}{rcr} x_1 & = & 5x_3 \\ x_2 & = & -2x_3 \\ x_3 & = & x_3 \end{array}\right.$ for $x_3 \in \mathbb{R}$. (b) $\left\{\begin{array}{rcr} x_1 & = & -4x_3 \\ x_2 & = & 3x_3 \\ x_3 & = & x_3 \end{array}\right.$ for $x_3 \in \mathbb{R}$.

24. Describe all solutions of the following homogeneous linear systems in parametric vector form, where the **coefficient matrix** is row equivalent to the given matrix.

a) $\begin{pmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{pmatrix}$

c) $\begin{pmatrix} 3 & -9 & 6 \\ -1 & 3 & -2 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{pmatrix}$

e) $\begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution: (a) $\begin{cases} x_1 = -9x_3 + 8x_4 \\ x_2 = 4x_3 - 5x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$ (b) $\begin{cases} x_1 = 5x_3 + 7x_4 \\ x_2 = -2x_3 + 6x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$

(c) $\begin{cases} x_1 = 3x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$ (d) $\begin{cases} x_1 = -3x_2 + 4x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$ (e) $\begin{cases} x_1 = 4x_2 - 5x_6 \\ x_2 = x_2 \\ x_3 = x_6 \\ x_4 = x_4 \\ x_5 = x_6 \\ x_6 = x_6 \end{cases}$

(f) $\begin{cases} x_1 = -5x_2 - 8x_4 - x_5 \\ x_2 = x_2 \\ x_3 = 7x_4 - 4x_5 \\ x_4 = x_4 \\ x_5 = x_5 \\ x_6 = 0 \end{cases}$

25. Suppose a nonhomogenous linear system ($Ax = b$) has a solution. Explain why the solution is unique precisely when the homogeneous linear system with the same coefficient matrix as the previous system ($Ax = 0$) has only the trivial solution.

Solution: A solution of $Ax = b$ is given by a particular solution of it plus a general solution of the homogeneous system. Thus the system $Ax = b$ has a unique solution if and only if $Ax = 0$ has a unique solution, that is, the homogenous system only has the trivial solution.

26. Suppose A is the 3×3 zero matrix, that is, with all zero entries. Describe the solution set of the homogeneous linear system with coefficient matrix A ($Ax = 0$).

Solution: The solution set is the entire $\mathbb{R}^3$.

27. Answer the following questions:

- does the equation $Ax = 0$ have a nontrivial solution?,
- does the equation $Ax = b$ have at least one solution for every non zero b?,

for the following coefficient matrices:

a) A is a 3×3 matrix with three pivot positions.

b) A is a 3×3 matrix with two pivot positions.

c) A is a 3×2 matrix with two pivot positions.

d) A is a 2×4 matrix with two pivot positions.

Solution: (a) No. Yes. (b) Yes. No. (c) No. No. (d) Yes. Yes.

28. Suppose an economy has only two sectors. Goods and Services. Each year, Goods sells 80 % of its output to Services and keeps the rest, while Services sells 70 % of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.

Solution: Let p_G be the price of goods and p_S be the price of services. The solution is $p_G = \dfrac{7}{8}p_S$.

29. Consider an economy with three sectors, Chemical & Metals, Fuels & Power, and Machinery. Chemicals sells 30 % of its output to Fuels and 50 % to Machinery and retains the rest. Fuels sells 80 % of its output to Chemicals and 10 % to Machinery and retains the rest. Machinery sells 40 % to Chemicals and 40 % to Fuels and retains the rest.

a) Construct the exchange table for this economy.

b) Develop a system of equations that leads to prices at which each sector's income matches its expenses. Then write the augmented matrix that can be row reduced to find these prices.

Solution:

(a)

Ch.& Met.	F. & P.	Mach.	
0,2	0,8	0,4	Ch.& Met.
0,3	0,1	0,4	F.& P.
0,5	0,1	0,2	Mach.

$$\text{(b)}\ \left\{\begin{array}{rrrcl} -0{,}8p_{ChMet} & +0{,}8p_{FP} & +0{,}4p_{Mach} & = & 0 \\ 0{,}3p_{ChMet} & -0{,}9p_{FP} & +0{,}4p_{Mach} & = & 0 \\ 0{,}5p_{ChMet} & +0{,}1p_{FP} & -0{,}8p_{Mach} & = & 0 \end{array}\right.$$

30. Balance the following chemical equation, that appear unbalanced, using linear systems.

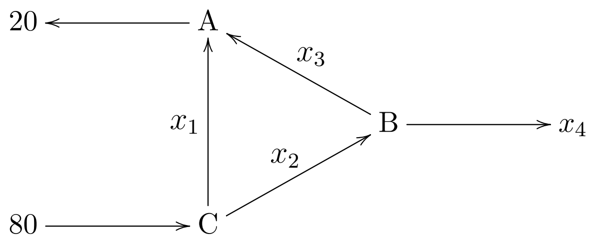
Boron sulfide reacts violently with water to form boric acid and hydrogen sulfide gas (the smell of rotten eggs).

$$\mathrm{B_2S_3} + \mathrm{H_2O} \to \mathrm{H_3BO_3} + \mathrm{H_2S}.$$

Solution: The systems of linear equations comes from equating the atoms in the chemical equation after adding some coefficients: $x_1\mathrm{B_2S_3} + x_2\mathrm{H_2O} \to x_3\mathrm{H_3BO_3} + x_4\mathrm{H_2S}$.

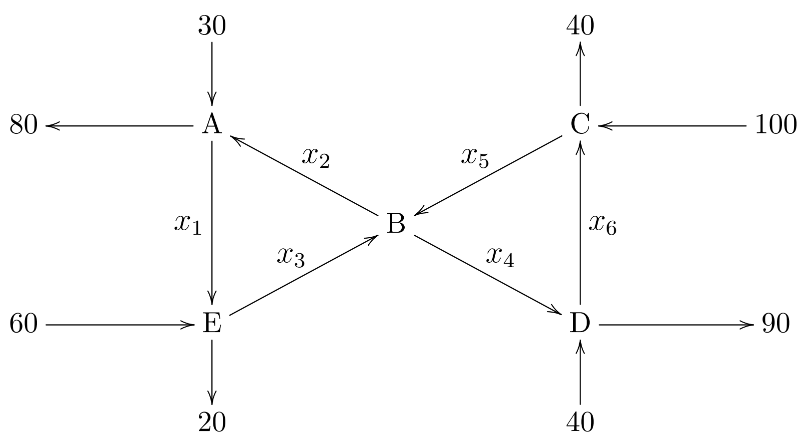
The result is $\dfrac{x_4}{3}\mathrm{B_2S_3} + 2x_4\mathrm{H_2O} \to \dfrac{2}{3}x_4\mathrm{H_3BO_3} + x_4\mathrm{H_2S}$ for any $x_4 \in \mathbb{R}$. In general x_4 is taking such that all the coefficients in the chemical equation are rational numbers, for instance, $x_4 = 3$.

31. Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the largest possible value for x_3?



Solution: $x_1 = 20 - x_3$, $x_2 = 60 + x_3$, $x_4 = 60$. As all the flows must be nonnegative, from $x_1 = 20 - x_3 \geq 0$, $x_2 = 60 + x_3 \geq 0$, we obtain that the largest possible value for x_3 is 20.

32. Find the general flow pattern in the network shown in the figure. Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by x_2, x_3, x_4 and x_5?



Solution: $x_1 = x_3 - 40$, $x_2 = x_3 + 10$, $x_4 = x_6 + 50$, $x_5 = x_6 + 60$. As all the flows must be nonnegative, the minimum flow in the branch x_2 is 50, in x_3 is 40, in x_4 is 50, in x_5 is 60, in x_6 is 0.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 1.1-1.2.**
- **Section 1.5.**
- **Section 1.6.**