Homework sheet 3: REAL VECTOR SPACES

(with solutions)

Year 2011-2012

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1. Let W be the set of all vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$, where $s,\ t \in \mathbb{R}$.

   Show that W is a subspace of $\mathbb{R}^4$.

   Solution: It consists of proving that $0 \in W$, $\begin{bmatrix} s_1+3t_1 \\ s_1-t_1 \\ 2s_1-t_1 \\ 4t_1 \end{bmatrix} + \begin{bmatrix} s_2+3t_2 \\ s_2-t_2 \\ 2s_2-t_2 \\ 4t_2 \end{bmatrix} \in W$ and $\lambda \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \in W$ for $s_1, s_2, s, t_1, t_2, t, \lambda \in \mathbb{R}$.

2. Let W be the set of all vectors of the form shown, where a, b and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

   a) $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$ b) $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$

   Solution: (a) is not a subspace because 0 does not belong to the subset. (b) is a subspace because it has $\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ as a spanning set.

3. Let W be the union of the first and third quadrants in the xy-plane. That is, let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$. If $u \in W$ and α is any scalar, is αu in W? Why? Find specific vectors u, v in W such that $u+v$ is not in W. This is enough to show that W is not a vector space.

   Solution: $\alpha u \in W$ because $(\alpha x)(\alpha y) = \alpha^2(xy) \geq 0$ since $\begin{bmatrix} x \\ y \end{bmatrix} \in W$. For example, $u = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in W$, $v = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \in W$, but $u+v \notin W$.

4. Show that the given set W is a vector space or find a specific example to the contrary.

a) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} a + 3b = c \\ b + c + a = d \end{matrix} \right\}$ b) $\left\{ \begin{bmatrix} b - 5d \\ 2b \\ 2d + 1 \\ d \end{bmatrix} : b, d \in \mathbb{R} \right\}$

Solution: (a) is a subspace because it is spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \right\}$. (b) is not a subspace because it does not contain 0.

5. Let $\mathbb{P}_n$ be the set of all polynomials of degree less or equal to n. Determine if the following subsets of $\mathbb{P}_n$ are subspaces or not.

   a) The set of all polynomials of the form $p(t) = at^2$, where $a \in \mathbb{R}$.

   b) The set of all polynomials of the form $p(t) = a + t^2$, where $a \in \mathbb{R}$.

Solution: (a) is a subspace since the three properties to be a subspace can be proved (check them). (b) is not a subspace because it does not contain the zero polynomial.

6. Determine if the set H of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of $\mathrm{M}_{2\times 2}$, where $a, b, d \in \mathbb{R}$.

Solution: It is a subspace.

7. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

   a) Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   Solution: $\mathbf{w}$ is not in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. There are only three vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

   b) How many vectors are in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   Solution: There are infinitely many vectors.

   c) Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

   Solution: Yes, it is because $\mathbf{v}_1 + (1 - 2c)\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{w}$ for any $c \in \mathbb{R}$.

   d) Is $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 14 \end{pmatrix}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   Solution: Yes, it is because $-5\mathbf{v}_1 + (3 - 2c)\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{u}$ for any $c \in \mathbb{R}$.

8. Determine if $w = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ is in $\mathrm{Nul}\, A$, where $A = \begin{pmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{pmatrix}$.

Solution: Yes, it is because $Aw = 0$.

9. Find an explicit expression, in terms of the generating vectors, of the null space of the following matrices:

$$\text{(a)} \begin{pmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: (a) $\text{span}\left\{ \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. (b) $\text{span}\left\{ \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

10. Find Nul$A$ for the matrix $A = \begin{pmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{pmatrix}$.

Solution: $\text{Nul}\, A = \text{span}\left\{ \begin{pmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$.

11. Assume that $A \sim B$, use this information to find Nul A and Col A.

$$A = \begin{pmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution: $\text{Col}\, A = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ -5 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 5 \\ -2 \end{pmatrix} \right\}$. $\text{Nul}\, A = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{pmatrix} \right\}$.

12. Find the matrix A such that $\text{Col}\, A = \left\{ \begin{pmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{pmatrix} \mid b, c, d \in \mathbb{R} \right\}$.

Solution: $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{pmatrix}$.

13. Let $A = \begin{pmatrix} -6 & 12 \\ -3 & 6 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Determine if $\mathbf{w}$ is in Col A and also in Nul A.

Solution: $\mathbf{w}$ is in $\operatorname{Col} A$ because $A\begin{pmatrix}(-1+6y)/3\\ y\end{pmatrix} = \mathbf{w}$ for any $y \in \mathbb{R}$. $\mathbf{w}$ is in $\operatorname{Nul} A$ because $A\mathbf{w} = 0$.

14. Determine if the following transformations are linear or not:

    a) $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$.
    b) $T(x_1, x_2) = (x_2, 0, x_1 + x_2, 0)$.
    c) $T(x_1, x_2) = (x_2, 1, x_1 + x_2, 0)$.
    d) $T(x_1, x_2, x_3) = (x_3, x_2)$.
    e) $T(x_1, x_2) = x_1 x_2$.
    f) $T(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

    Solution: (a), (c) and (e) are not linear. (b), (d) and (f) are linear.

15. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x) = Ax$ where $A = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$. Find the images under T of $u = \begin{pmatrix}1\\ -3\end{pmatrix}$ and $v = \begin{pmatrix}a\\ b\end{pmatrix}$.

    Solution: $T(u) = \begin{pmatrix}2\\ -6\end{pmatrix}$, $T(v) = \begin{pmatrix}2a\\ 2b\end{pmatrix}$.

16. Let A be a 6×5 matrix. What must be a and b in order to define $T\colon \mathbb{R}^a \to \mathbb{R}^b$ by $T(x) = Ax$?

    Solution: $a = 5$, $b = 6$.

17. How many rows and columns must a matrix A have in order to define a mapping T from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(x) = Ax$?

    Solution: $A \in M_{7\times 5}$, that is, A has 7 rows and 5 columns.

18. Let T be a linear transformation given by $T(x) = Ax$. Find x whose image under T is b, where

$$A = \begin{bmatrix}1 & 2 & -1\\ 2 & 3 & 1\\ 0 & 3 & -8\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}5\\ 9\\ 4\end{bmatrix}.$$

    Solution: $x = \begin{bmatrix}-2\\ 4\\ 1\end{bmatrix}$.

19. Let

$$A = \begin{bmatrix}1 & 2 & -7 & 5\\ 0 & 1 & -4 & 0\\ 1 & 0 & 1 & 6\\ 2 & -1 & 6 & 8\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}9\\ 5\\ 0\\ -9\end{bmatrix}.$$

a) Find all the vectors $x \in \mathbb{R}^4$ sucht that are mapped to zero under the transformation $x \to Ax$

b) Is b in the range of the linear transformation defined by A?

Solution: (a) $\ker A = \operatorname{Nul} A = \operatorname{Gen}\left\{\begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}\right\}$. (b) Yes, because $T\left(\begin{bmatrix} -6-c \\ 5+4c \\ c \\ 1 \end{bmatrix}\right) = b$ for any $c \in \mathbb{R}$.

20. Use a rectangular coordinate system to plot

$$u = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

and their images under the given transformation T defined by the rule $T(x) = Ax$. Describe geometrically what T does to each vector $x \in \mathbb{R}^2$.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution: The first matrix is the orthogonal projection with respect to axis OY. The second matrix is the axis symmetry about axis OY. The third matrix scales by two the first component of the vectors.

21. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation that maps $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into $\begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$. Use the fact that T is linear to find the images under T of $3u$, $-2v$ and $3u - 2v$.

Solution: $\begin{pmatrix} 21 \\ -9 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 0 \\ -8 \end{pmatrix}$, $\begin{pmatrix} 17 \\ -9 \\ -5 \end{pmatrix}$, respectively.

22. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a) Find the matrix associated with that transformation and provide the analytical expression of T.

b) Compute $T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

c) Compute $T^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

d) Is T injective? Is it onto?

Solution: (a) $\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$. (b) $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$. (c) $\begin{pmatrix} 11/2 \\ 3/2 \end{pmatrix}$. (d) It is injective and onto.

23. Determine whether the following set of vectors are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

a) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

b) $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -8 \\ 5 \\ 4 \end{pmatrix}.$

c) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$

Solution: (a) is not a basis, it is not a set of linearly independent vectors and it does not span $\mathbb{R}^3$. (b) is a basis, then it is a set of linearly independent vectors and it spans $\mathbb{R}^3$. (c) is not a basis, it is not a set of linearly independent vectors, but it spans $\mathbb{R}^3$.

24. Find a basis for the null spaces of the following matrix $\begin{pmatrix} 1 & 1 & -2 & 1 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -8 & 0 & 16 \end{pmatrix}$.

Solution: A basis for $\operatorname{Nul} A$ is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

25. Find a basis for $\operatorname{Span}\{v_1, \ldots, v_5\}$ if

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 6 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 5 \\ -3 \\ 3 \\ -4 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}.$$

Solution: A basis for $\operatorname{span}\{v_1, \ldots, v_5\}$ is $\{v_1, v_2, v_3\}$. Hint: identify the pivot columns of the matrix that consists of the columns $\{v_1, \ldots, v_5\}$.

26. Decide if the following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}.$$

Solution: Yes, they are. They are linearly independent because, for instance, $\begin{vmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & -6 \end{vmatrix} = -24 \neq 0$. We can only use the determinant to determine if a set of vectors is linearly independent or not if the matrix made of those vectors is a square matrix, otherwise the row echelon form of the matrix must be found.

27. Decide if the columns in the following matrix are linearly independent:

$$A = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & 1 & 3 \end{bmatrix}.$$

Solution: They are not linearly independent because, for instance, $\det A = 0$. The determinant can be computed using the cofactor expansion across a row or a column of the matrix.

28. Let

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 9 \\ -1 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} -2 \\ -6 \\ h \end{bmatrix}.$$

For what values of h

a) $v_3 \in \text{Span}[v_1, v_2]$?

b) v_1, v_2 and v_3 are linearly independent?

Solution: (a) For any value of h, $v_3 \in \text{span}[v_1, v_2]$ because $\frac{-2+3h}{10} v_1 - \frac{6+h}{10} v_2 = v_3$. (b) v_1, v_2 and v_3 are never linearly independent for any value of h.

29. Find a basis for the set of vectors in $\mathbb{R}^2$ in the line $y = 5x$.

Solution: A basis for such a set is $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

30. Let $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \\ -2 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 5 \\ -6 \\ -14 \end{pmatrix}$. It can be verified that $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = 0$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (Remark: The solution is not unique).

Solution: Possible bases are $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$. Hint: Use the spanning set theorem.

31. Fill in the blank space in the following sentence: "If A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if A has ________________ pivot columns".

Solution: n.

32. Find a basis and the dimension of the following subspaces:

a) $\left\{ \begin{pmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{pmatrix} \ : \ a,b,c \in \mathbb{R} \right\}.$

b) $\left\{ \begin{pmatrix} 2a \\ -4b \\ -2a \end{pmatrix} \ : \ a,b \in \mathbb{R} \right\}.$

c) $\left\{ \begin{pmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{pmatrix} \ : \ p,q \in \mathbb{R} \right\}.$

d) $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \ : a-3b+c=0,\ b-2c=0,\ 2b-c=0 \right\}.$

e) $\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \ : a-3b+c=0 \right\}.$

Solution: (a) has dimension 2. A basis is $\left\{ \begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} \right\}.$

(b) has dimension 2. A basis is $\left\{ \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} \right\}.$

(c) has dimension 2. A basis is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$

(d) has dimension 0. Hence, there is no a basis.

(e) has dimension 3. A basis is $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

33. Find the dimension of the subspace spanned by the vectors:

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix}.$$

Solution: It has dimension 3.

34. Determine the dimension of the subspaces Nul A and Col A in the following cases:

$a)\ A = \begin{pmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 6 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

$b)\ A = \begin{pmatrix} 3 & 2 \\ -6 & 5 \end{pmatrix}.$

$c)\ A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Solution: (a) $\dim \operatorname{Col} A = 4$, $\dim \operatorname{Nul} A = 3$. (b) $\dim \operatorname{Col} A = 2$, $\dim \operatorname{Nul} A = 0$. (c) $\dim \operatorname{Col} A = 2$, $\dim \operatorname{Nul} A = 1$.

35. Assume that the matrices A and B are row equivalent. Without additional computations determine the rank of A, the dimension of $\ker A$. Find a basis of $\operatorname{Col} A$ and $\ker A$.

$$a)\ A = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$b)\ A = \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 6 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution: (a) $\operatorname{rank} A = 3$, $\dim \ker A = 2$. A basis for $\operatorname{Col} A$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -3 \\ 0 \end{pmatrix} \right\}$.

A basis for $\ker A$ is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(b) $\operatorname{rank} A = 5$, $\dim \ker A = 1$. A basis for $\operatorname{Col} A$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 6 \\ 6 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 6 \\ 6 \\ -1 \end{pmatrix} \right\}$.

A basis for $\ker A$ is $\left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

36. If a 7×5 matrix A has rank 2, find the dimension of $\mathrm{Nul}\, A$ and the rank of A^T.

Solution: $\dim \mathrm{Nul} A = 3$, $\mathrm{rank}\, A^T = 2$.

37. Assume that a 6×8 matrix A has 4 pivots. What is the dimension of $\mathrm{Nul}\, A$? Is $\mathrm{Col}\, A = \mathbb{R}^4$? Justify your answers.

Solution: $\dim \mathrm{Nul} A = 4$. $\mathrm{Col}\, A \neq \mathbb{R}^4$ because $\mathrm{Col}\, A$ is a subspace of $\mathbb{R}^6$.

38. If the null space of a 8×7 matrix A has dimension 5, what is the dimension of the column space of A?

Solution: $\dim \mathrm{Col} A = 2$.

39. Let A be a 7×5 matrix. What is the largest possible rank of A? And if A is a 5×7 matrix? Justify your answers.

Solution: The largest possible for the rank of A is 5. The same for a 5×7 matrix A.

40. Let A be a 7×5 matrix. What is the smallest possible dimension of the null space of A?

Solution: The smallest possible dimension of $\mathrm{Nul}\, A$ is 0.

41. Assume that a nonhomogeneous system $Ax = b$ with 6 equations and 8 unknowns has 2 free variables. Is it possible to change some of the values in the column vector b so that the system is inconsistent?

Solution: No, it is not possible because the coefficient matrix always has six pivots, as many as rows. Hence, there is no way the last column of the augmented matrix is a pivot column so that the system was inconsistent.

42. Let A be a $m \times n$ matrix and $b \in \mathbb{R}^m$. What are the relative rank values of the matrices A and $\begin{pmatrix} A & b \end{pmatrix}$ so that the system $Ax = b$ is consistent? (*Rouché-Frobenius' Theorem*).

Solution: $\mathrm{rank}\, A = \mathrm{rank}\, (A \,|\, b)$.

43. Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be bases for a vector space V and suppose $b_1 = -2c_1 + 4c_2$, $b_2 = 3c_1 - 6c_2$. Find the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$. Find $[x]_{\mathcal{C}}$ for $x = 2b_1 + 3b_2$.

Solution: $P_{\mathcal{CB}} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}$, $[x]_{\mathcal{C}} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$.

44. Let $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{D} = \{d_1, d_2, d_3\}$ be bases for a vector space V and suppose $P = [[d_1]_{\mathcal{A}}\, [d_2]_{\mathcal{A}}\, [d_3]_{\mathcal{A}}]$. Which of the following equations is satisfied by P for all $x \in V$?

$$\text{(a) } [x]_{\mathcal{A}} = P\,[x]_{\mathcal{D}} \quad \text{(b) } [x]_{\mathcal{D}} = P\,[x]_{\mathcal{A}}$$

Solution: (a).

45. Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $\mathcal{F} = \{f_1, f_2, f_3\}$ be bases for a vector space V and suppose $f_1 = 2d_1 - d_2 + d_3$, $f_2 = 3d_2 + d_3$, $f_3 = -3d_1 + 2d_3$. Find the change-of-basis matrix from $\mathcal{F}$ to $\mathcal{D}$. Find $[x]_\mathcal{D}$ for $x = f_1 - 2f_2 + 2f_3$.

Solution: $P_{\mathcal{D}\mathcal{F}} = \begin{pmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, $[x]_\mathcal{C} = \begin{pmatrix} -4 \\ -7 \\ 3 \end{pmatrix}$.

46. Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. Find the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$ if

$$a)\ b_1 = \begin{pmatrix} -1 \\ 8 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ -7 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$b)\ b_1 = \begin{pmatrix} 6 \\ -12 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 3 \\ 9 \end{pmatrix}.$$

Solution: (a) $P_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} 9 & -8 \\ -10 & 9 \end{pmatrix}$, $P_{\mathcal{B}\mathcal{C}} = \begin{pmatrix} 9 & 8 \\ 10 & 9 \end{pmatrix}$. (b) $P_{\mathcal{C}\mathcal{B}} = \begin{pmatrix} 3/5 & 1 \\ -2 & 0 \end{pmatrix}$, $P_{\mathcal{B}\mathcal{C}} = \begin{pmatrix} 0 & -1/2 \\ 1 & 3/10 \end{pmatrix}$.

47. Find the coordinate vectors $[x]_\mathcal{B}$ of x in the basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ in the following cases:

$$a)\ b_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad x = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

$$b)\ b_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

Solution: (a) $[x]_\mathcal{B} = P_{\mathcal{B}\mathcal{B}_c}[x]_{\mathcal{B}_c} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$. (b) $[x]_\mathcal{B} = P_{\mathcal{B}\mathcal{B}_c}[x]_{\mathcal{B}_c} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$.

48. Find the change-of-basis matrix, $P_{\mathcal{B}_c,\mathcal{B}}$ from bases $\mathcal{B}$ to the standard bases $\mathcal{B}_c$ of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively:

$$\text{(a) } \mathcal{B} = \left\{ \begin{pmatrix} 2 \\ -9 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \end{pmatrix} \right\}, \quad \text{(b) } \mathcal{B} = \left\{ \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 8 \\ -2 \\ 7 \end{pmatrix} \right\}.$$

Solution: (a) $P_{\mathcal{B}_c,\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ -9 & 8 \end{pmatrix}$. (b) $P_{\mathcal{B}_c,\mathcal{B}} = \begin{pmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{pmatrix}$.

49. The vectors $v_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, $v_3 = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$ span $\mathbb{R}^2$, but do not form a basis. Find two different ways to express vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as a linear combination of v_1, v_2 and v_3.

Solution: For example $5v_1 - 2v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $10v_1 - 3v_2 + v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Other possible solutions come from giving real values to c in the following expression: $(5+5c)v_1 + (-2-c)v_2 + cv_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 4.1-4.2**
- **Sections 1.8-1.9.**
- **Sections 4.3-4.7.**