Homework sheet 4: EIGENVALUES AND EIGENVECTORS. DIAGONALIZATION

(with solutions)

Year 2011-2012

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1. Is $\lambda = -3$ an eigenvalue of $\begin{pmatrix} -1 & 4 \\ 6 & 9 \end{pmatrix}$? Why or why not?

   Solution: Yes, it is because it satisfies $\det(A - (-3)\mathrm{I}_2) = 0$.

2. Is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}$? If so, find the eigenvalue.

   Solution: Yes, it is because $\begin{pmatrix} 5 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The eigenvalue of the eigenvector is 3.

3. Is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix}$? If so, find the eigenvalue.

   Solution: It is not an eigenvector because $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 1 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$.

4. Find a basis for the eigenspace corresponding to each listed eigenvalue:

   a) $A = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}$, $\lambda = -5$.

   b) $A = \begin{pmatrix} 4 & 1 \\ 3 & 6 \end{pmatrix}$, $\lambda = 3, 7$.

   c) $A = \begin{pmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{pmatrix}$, $\lambda = 3$.

   Solution: (a) $V(-5) = \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$.

   (b) $V(3) = \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, $V(7) = \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$.

   (c) $V(3) = \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$.

5. Find the eigenvalues and eigenvectors of the following matrices:

$$\text{(a)} \begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{(d)} \begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}$$

Solution: (a) $V(3) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, $V(4) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} -7 \\ 6 \\ 1 \end{pmatrix} \right\}$.

(b) $V(4) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix} \right\}$, $V(0) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, $V(-3) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(c) $V(2) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$, $V(4) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, $V(-1) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \right\}$.

(d) $V(1) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 3 \\ -6 \\ -8 \end{pmatrix} \right\}$, $V(7) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \right\}$, $V(-4) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \right\}$.

6. List the real eigenvalues, repeated according to their multiplicities, of the following matrices:

$$\text{(a)} \begin{pmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 2 & 3 & 3 & -5 \end{pmatrix}$$

Solution: (a) 5 has multiplicity of 2, 2 and 3 have both multiplicity 1.

(b) 3, 2, 6, -5. All have multiplicity 1.

7. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional.

$$A = \begin{pmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution: $h = -3$.

8. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. (*Hint:* Suppose a nonzero x satisfies $Ax = \lambda x$.)

Solution: As λ is an eigenvalue of an invertible matrix A , the eigenvalue λ is nonzero and there exist a nonzero vector x such that $Ax = \lambda x$. Multiply the equation by $\lambda^{-1}A^{-1}$,

$$\lambda^{-1}A^{-1}Ax = (\lambda^{-1}\lambda)A^{-1}x; \quad \lambda^{-1}x = A^{-1}x.$$

Thus λ^{-1} is an eigenvalue of A^{-1}.

9. Let $A = PDP^{-1}$. Compute A^4 if $P = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

   Solution: $A^4 = PD^4P^{-1} = \begin{pmatrix} 226 & -325 \\ 90 & -209 \end{pmatrix}$.

10. Use the factorization $A = PDP^{-1}$ to compute A^k, where k represents an arbitrary positive integer, if

$$A = \begin{pmatrix} a & 0 \\ 2(a-b) & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

   Solution: $A^k = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ 2(a^k - b^k) & b^k \end{pmatrix}$.

11. The matrix A is factored in the form PDP^{-1}. Find the eigenvalues of A and a basis for each eigenspace.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -3 & 1 & 9 \\ -1 & 0 & 3 \end{pmatrix}$$

   Solution: The eigenvalues are 3 (with multiplicity 2) and 4. A basis for the eigenspace of eigenvalue 3 is given by $\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}$ and a basis for the eigenspace of eigenvalue 4 is given by $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

12. Diagonalize the following matrices and give the matrices P and D in the factorization $A = PDP^{-1}$, if possible.

   a) $\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

   b) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

   c) $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

   d) $\begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

   e) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$

$f)$ $\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

Solution: (a) It is not diagonalizable because there is only one eigenvalue, 3, and it only has one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ associated with.

(b) It is diagonalizable because it has two different eigenvalues: 5 and -2. $D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}$, $P = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}$.

(c) $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(d) $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(e) There is only one eigenvalue 0 which has multiplicity 1. This matrix is not diagonalizable in $\mathbb{R}$.

(f) It is diagonalizable because the matrix has 3 different eigenvalues. $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

13. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

    Solution: No, A is not diagonalizable because we cannot have a basis of eigenvectors for $\mathbb{R}^3$.

14. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspace is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

    Solution: If the remaining eigenspace has dimension strictly less than 2, then A is not diagonalizable because we cannot construct a basis of eigenvectors for $\mathbb{R}^7$.

15. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

    Solution: If A is invertible, all the eigenvalues are nonzero. If A is diagonalizable, there exist matrices P and D such that $A = PDP^{-1}$. Take the inverse of this equality,

    $$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}.$$

    Thus A^{-1} is diagonalizable.

16. Determine if the application matrix of the following linear transformations is diagonalizable.

    a) $F(x,y) = (2y, x-y)$,

    b) $G(x,y) = (2x-y, x)$,

    c) $H(x,y,z) = (2(x+z), x-z, x+3z)$.

    Solution: (a) The matrix $\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ is diagonalizable with $P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.

    (b) The matrix $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable because it only has one eigenvalue, 1, and this eigenvalue has only one eigenvector associated with.

    (c) The matrix $\begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 0 & 3 \end{pmatrix}$ is diagonalizable with $P = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

17. Determine a basis for $\mathbb{R}^3$ such that the application matrix of the above linear transformation $H(x,y,z) = (2(x+z), x-z, x+3z)$ in that basis is diagonal.

    Solution: In the basis given by the columns of P in 16(c), the application matrix of H is just D.

Additional exercises: D. C. Lay "Linear algebra and its applications", 2012.

- **Sections 5.1-5.4.**